Polylogarithms and motivic Galois groups

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This paper is an enlarged version of the lecture given at the AMS conference "Motives" in Seattle, July 1991. More details can be found in [G2].

My aim is to formulate a precise conjecture about the structure of the Galois group Gal $(\mathcal{M}_T(F))$ of the category $\mathcal{M}_T(F)$ of mixed Tate motivic sheaves over Spec F, where F is an arbitrary field. This conjecture implies (and in fact is equivalent to) a construction of complexes $\Gamma(F, n)_{\mathbb{Q}}$ that should satisfy all the Beilinson-Lichtenbaum axioms modulo torsion.

In particular, we get a hypothetical description of $K_n(F) \otimes \mathbb{Q}$ by generators and relations that generalize the definition of Milnor's K-groups. In the case when F is a number field this together with the Borel theorem implies

Zagier's conjecture [Z1]: the value of the Dedekind zeta-function $\zeta_F(s)$ of an arbitrary number field F at the point n is expressed by a determinant whose entries are rational linear combinations of values of the **classical** n-logarithms at (complex embedding of) some elements of this field.

In §3 I give a proof of Zagier's conjecture in the case $n = 3$. The Invented by Euler classical polylogarithms are defined on the unit disc $|z| \leq 1$ by absolutely convergent series

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{n^k} \ .$$

They can be continued analytically to a multivalued function on $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$. Their properties including the differential and functional equations play the key role in all our considerations. However the special role of the *projective line* and classical polylogarithms in the theory of mixed Tate motives remains absolutely mysterious. Formulas that led me to the conjectures about $\Gamma(F, n)_{\mathbb{Q}}$ and $Gal(\mathcal{M}_T(F))$ are discussed in §4.

In §5 I will construct explicitly a regulator map r_3 from the motivic complex $\Gamma(X; 3)_{\mathbb{Q}}$ attached to any algebraic variety over $\mathbb{C}$ to the third Deligne complex of $X(\mathbb{C})$. (For a generalization of this construction to motivic complexes $\Gamma(X; n)_{\mathbb{Q}}$ see [G3]). Then an explicit formula for the universal motivic

Chern class $c_3 \in H^6_{\mathcal{M}}(BGL_3(F)_\bullet, \mathbb{Q}(3))$ will be given. Applying the regulator we get a realization of c_3 in the real Deligne cohomology. I need the last result in order to complete the proof of Zagier's conjecture.

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1 Conjectures

First of all I need to explain how to think about Gal $(\mathcal{M}_T(F))$. So for convenience of the reader I reproduce basic definitions from [B-D].

1.1 Mixed Tate Categories. ([B-D], see also [BMS], [B2], [D2]). A mixed Tate category is a Tannakien $\mathbb{Q}$-category $\mathcal{M}$ together with a fixed invertible object $\mathbb{Q}(1)_{\mathcal{M}}$ such that

a) Any simple object in $\mathcal{M}$ is isomorphic to

$$\mathbb{Q}(m)_{\mathcal{M}} := \mathbb{Q}(1)^{\otimes m}_{\mathcal{M}} , \quad m \in \mathbb{Z} .$$

b) $\dim \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(o)_{\mathcal{M}}, \mathbb{Q}(m)_{\mathcal{M}}) = \delta_{o,m}$

$$\mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Q}(o)_{\mathcal{M}}, \mathbb{Q}(m)_{\mathcal{M}}) = 0 \quad \text{for } m \leq 0 .$$

(I recall that "Tannakien" means in particular that there is a $\otimes$-product in $\mathcal{M}$; the function $\mathcal{F} \mapsto \mathcal{F} \otimes \mathbb{Q}(1)_{\mathcal{M}}$ is an equivalence of categories).

A Tate functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ between mixed Tate categories is an exact $\otimes$ functor such that $F(\mathbb{Q}(1)_{\mathcal{M}_1}) = \mathbb{Q}(1)_{\mathcal{M}_2}$. Sometimes I will write $\mathbb{Q}(m)$ instead of $\mathbb{Q}(m)_{\mathcal{M}}$.

An object $\mathcal{F}$ of $\mathcal{M}$ has a canonical finite increasing filtration $\subset \mathcal{F}_{\leq i} \subset \mathcal{F}_{\leq i+1} \subset \ldots$ such that $\mathcal{F}_i := \mathcal{F}_{\leq i}/\mathcal{F}_{\leq i-1}$ is isomorphic to a direct sum of $\mathbb{Q}(-i)$'s. There is a canonical fiber functor to the category of finite dimensional graded $\mathbb{Q}$-vector spaces $\omega_{\mathcal{M}} : \mathcal{M} \to \mathrm{Vect}^\bullet_{\mathbb{Q}}$:

$$\omega_{\mathcal{M}}(\mathcal{F}_i) := \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(-i), \mathcal{F}_i) , \quad \omega_{\mathcal{M}}(\mathcal{F}) := \oplus_i \omega_{\mathcal{M}}(\mathcal{F})_i .$$

Let $L(\mathcal{M})$ be the space of all derivations of $\omega_{\mathcal{M}}$: an element $\varphi \in L(\mathcal{M})$ is a natural endomorphism of the functor $\omega_{\mathcal{M}}$ such that $\varphi_{\mathcal{F}\otimes G} = \varphi_{\mathcal{F}} \otimes \mathrm{id}_{\omega(G)} +$

$\mathrm{id}_{\omega(\mathcal{F})} \otimes \varphi_G$. Then $L(\mathcal{M})$ is canonically equipped with the structure of a graded pro-Lie algebra: $L(\mathcal{M}) = \oplus L(\mathcal{M})_i$, where

$$F(\mathcal{M})_i := \{\varphi \in L(\mathcal{M}) | \varphi(\mathcal{F}) : \omega_{\mathcal{M}}(\mathcal{F})_\bullet \to \omega_{\mathcal{M}}(\mathcal{F})_{\bullet+i}\} .$$

(Recall that "graded pro-Lie algebra" is a projective limit of finite dimensional Lie algebras) It is easy to prove that $L(\mathcal{M})_i = 0$ for $i \geq 0$. Such Lie algebras are called mixed Tate pro-Lie algebras. For any mixed Tate Lie algebra L the category L-mod of finite dimensional graded continuous L-modules is a mixed Tate category. The object $\mathbb{Q}(1)$ in this category is a trivial one dimensional L-module placed in degree -1; the fiber functor $\omega : L\text{-mod} \longrightarrow \mathrm{Vect}^\bullet_{\mathbb{Q}}$ is just forgetting of L-action functor. For any mixed Tate category $\mathcal{M}$ the fiber functor $\omega_{\mathcal{M}}$ lifts canonically to the Tate functor $\omega_{\mathcal{M}} : \mathcal{M} \to L(\mathcal{M})$-mod. It is easy to prove that $\omega_{\mathcal{M}}$ is an equivalence of categories. Note that any Tate functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ commutes with ω's and so defines the morphism $F_\bullet : L(\mathcal{M}_1)_\bullet \to L(\mathcal{M}_2)_\bullet$ of corresponding mixed Tate algebras. For an object $\mathcal{F} \in \mathcal{M}$

$$H^\bullet_{\mathcal{M}}(\mathcal{F}) := \mathrm{Ext}^\bullet_{\mathcal{M}}(\mathbb{Q}(o), \mathcal{F}) = H^\bullet(L(\mathcal{M})_\bullet, \omega_{\mathcal{M}}(\mathcal{F})) \tag{1}$$

Remark. Let $G(\mathcal{M})$ be a prounipotent group with the Lie algebra $L(\mathcal{M})$. Note that $G(\mathcal{M})$ acts on any continuous $L(\mathcal{M})$-module. There is a semidirect product $G_m \times G(\mathcal{M})$ where G_m is the multiplicative group and the action of G_m on $G(\mathcal{M})$ provides the grading on $L(\mathcal{M})$-modules. So the category of finite dimensional graded continuous $L(\mathcal{M})$-modules is canonically isomorphic to the category of $G_m \times G(\mathcal{M})$ finite dimensional continuous modules.

1.2 The motivic Lie algebra $L(F)_\bullet$. A.A. Beilinson conjectured ([B1]) that for arbitrary field F there exists a mixed Tate category $\mathcal{M}_T(F)$ of mixed motivic Tate sheaves over Spec F such that

$$\mathrm{Ext}^i_{\mathcal{M}_T(F)}(\mathbb{Q}(o), \mathbb{Q}(m)) \cong gr^n_\gamma K_{2n-i}(F)_{\mathbb{Q}} \tag{2}$$

where γ is the γ-filtration on K-groups (see [So]) and for an ablian group A we put $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. Let $L(F)_\bullet = \oplus_{n=1}^\infty L(F)_{-n}$ be the corresponding mixed Tate Lie algebra. Its cohomology $H^i(L(F)_\bullet)$ has a natural grading by positive integers because $L(F)_\bullet$ itself is a negatively graded Lie algebra. Let us denote by $H^i_{(n)}(L(F)_\bullet)$ the part of degree n with respect to this grading. Then axiom (1.2) means that

$$H^i_{(n)}(L(F)_\bullet) = gr^n_\gamma K_{2n-i}(F)_{\mathbb{Q}} . \tag{3}$$

Moreover, this isomorphism should be compatible with natural products on $H^*(L(F)_\bullet)$ and $K_*(F)$. It also should be functorial with respect to embeddings of fields $i : F \hookrightarrow E$. (More precisely the corresponding morphism of schemes $\tilde{i} : \text{Spec } E \to \text{Spec } F$ should lift to a morphism of mixed Tate categories $\tilde{i}^* : \mathcal{M}_T(F) \to \mathcal{M}_T(E)$ commuting with the fiber functors, and so provides us a homomorphism of the Lie algebras $\tilde{i}_\bullet : L(E)_\bullet \to L(F)_\bullet$). The Galois group $\text{Gal}(\mathcal{M}_T(F))$ is by definition the semidirect product $G_m \times G(\mathcal{M}_T(F))$ (see above).

This conjecture gives a new point of view on algebraic K- theory. Let me give some examples demonstrating how powerful it is.

Example 1.1 $H^i(L(F)_\bullet) = 0$ for $i < 0$ and $H^0_{(n)}(L(F)_\bullet) = 0$ for $n > 0$. So $gr^n_\gamma K_m(F)_\mathbb{Q} = 0$ for $m \geq 2n > 0$. But this is just Beilinson-Soulé conjecture.

Example 1.2 The degree n part of the cochain complex $(\Lambda^\bullet(L(F)^\vee_\bullet), \partial)$ of the Lie algebra $L(F)_\bullet$ forms a subcomplex $(\Lambda^\bullet_{(n)}(L(F)^\vee_\bullet, \partial)$:

$$L^\vee_{-n} \xrightarrow{\partial} \dots \xrightarrow{\partial} L^\vee_{-2} \otimes \Lambda^{n-2} L^v_{-1} \xrightarrow{\partial} \Lambda^n L^\vee_{-1} \tag{4}$$

(we write L_{-n} instead of $L(F)_{-n}$). In particular it is concentrated in degrees $[1, n]$. $(\Lambda^m_{(n)}(L(F)^\vee_\bullet) = 0$ for $m > n$ because $L(F)_\bullet$ is graded by strictly negative integers). So according to (1.3) $gr^n_\gamma K_m(F)_\mathbb{Q} = 0$ for $m > n$. This is a well known theorem in K-theory that follows from results of A.A. Suslin [S1] (see [So].)

Example 1.3 (Relation with Milnor K-theory). Applying (1.3) in the simplest case $i = n = 1$ we get

$$H^1_{(1)}(L(F)_\bullet) \stackrel{\text{def}}{=} L(F)^\vee_{-1} \stackrel{(1.3)}{=} K_1(F)_\mathbb{Q} = F^*_\mathbb{Q} \ . \tag{5}$$

Here $W \longrightarrow W^\vee$ is the duality between $\varprojlim$ and $\varinjlim$ of finite dimensional $\mathbb{Q}$-vector spaces: $(W^\vee)^\vee = W$. The structure of an $\varinjlim$ of finite dimensional $\mathbb{Q}$-vector space on $F^*_\mathbb{Q}$ is defined as follows. Let $\mathbb{Z}[P^1_F]$ is the free abelian group generated by symbols $\{x\}$ where x runs all F-points of the projective line P^1. Let us denote by $\mathcal{R}_1(F)$ the subgroup generated by symbols $\{\infty\}, \{0\}, \{xy\} - \{x\} - \{y\} \quad (x, y \in F^*)$. Then there is canonical isomorphism

$$\mathbb{Z}[P^1_F]/\mathcal{R}_1(F) \to F^*; \quad \{x\} \mapsto x; \{\infty\}, \{0\} \mapsto 1 \ .$$

Both $\mathbb{Q}[P^1_F] := \mathbb{Z}[P^1_F] \otimes \mathbb{Q}$ and $\mathcal{R}_1(F)_\mathbb{Q}$ are $\varinjlim$ of finite dimensional $\mathbb{Q}$-vector spaces, so we get the same structure on $F^*_\mathbb{Q}$.

Now look at the degree 2 part of the cochain complex of $L(F)$. (We use (1.5)):

$$L^{\vee}_{-2} \stackrel{\partial}{\longrightarrow} \Lambda^2 F^*_{\mathbb{Q}} \ .$$

According to (1.3) Coker $\partial = K_2(F)_{\mathbb{Q}}$. So by Matsumoto-Moore theorem Im∂ is generated by symbols $(1-x)\wedge x$. Hence we get a homomorphism of complexes, where $\delta : \{x\} \mapsto (1-x)\wedge x$

$$\begin{array}{ccc} Q[P^1_F] & \stackrel{\delta}{\longrightarrow} & \wedge^2 F^*_{\mathbb{Q}} \\ \downarrow & & \| \\ L(F)^{\vee}_{-2} & \stackrel{\partial}{\longrightarrow} & \wedge^2 F^*_{\mathbb{Q}} \end{array}$$

Further,

$$\partial(L^{\vee}_{-2} \otimes \wedge^{n-2} L^{\vee}_{-1}) = \partial(L^{\vee}_{-2}) \wedge \wedge^{n-2} L^{\vee}_{-1} \ ,$$

so

$$H^n_{(n)}(L(F)_\bullet) = K^M_n(F)_{\mathbb{Q}} := \frac{\wedge^n F^*}{(1-x)\wedge x \wedge \wedge^{n-2} F^*} \otimes \mathbb{Q} \ .$$

(Here $K^M_*(F)$ is the Milnor ring of the field F (see [M])). Comparing with (1.3) we obtain $gr^n_\gamma K_n(F)_{\mathbb{Q}} = K^M_n(F)_{\mathbb{Q}}$. More precisely we get the following: multiplication in $K_*(F)$ induces a map $m : K_1(F) \times \ldots \times K_1(F) \to K_n(F)$ that factorizes through a map $s : K^M_n(F) \to K_n(F)$

$$\begin{array}{ccc} F^* \times \ldots \times F^* & \stackrel{m}{\longrightarrow} & K_n(F) \\ & \searrow \quad \nearrow_{s} & \\ & K^M_n(F) & \end{array}$$

Then the composition $K^M_n(F) \to K_n(F) \to gr^n_\gamma K_n(F)$ is an isomorphism modulo torsion. But this is the well known theorem of A.A. Suslin [S1]. (In fact Suslin proved that it is an isomorphism modulo $(n-1)!$.)

Example 1.4 Complexes $(\wedge^\bullet_{(n)}(L(F)^{\vee}_\bullet), \partial)$ should satisfy the Beilinson-Lichtenbaum axioms modulo torsion (see [B1] and [L1]).

More precisely, the (hypothetical) properties of the Lie algebra $L(F)_\bullet$ provides most of all axioms: these complexes concentrated in degrees $[1, n]$

by definition; relation with algebraic K-theory given by (1.3); the DGA structure of $\wedge^\bullet(L(F)^\vee_\bullet)$ gives a morphism of complexes

$$(\wedge^\bullet_{(n)}(L(F)^\vee_\bullet), \partial) \otimes (\wedge^\bullet_{(m)}(L(F)^\vee_\bullet), \partial) \to (\wedge^\bullet_{(n+m)}(L(F)^\vee_\bullet, \partial)$$

and example 1.3 shows that

$$H^n(\wedge^\bullet_{(n)}(L(F)^\vee_\bullet)) = K^M_n(F)_\mathbb{Q} .$$

The only axiom that remains unclear from this point of view is the existence of the transfer

$$(\wedge^\bullet_{(n)}(L(E)^\vee_\bullet)) \to (\wedge^\bullet_{(n)}(L(F)^\vee_\bullet))$$

for a finite extension of fields $F \subset E$. On the other hand, if we know something about $K_*(F)_\mathbb{Q}$, then conjecture (1.3) provides us some information about the structure of the Lie algebra $L(F)_\bullet$.

Example 1.5 Let F be a number field. Then it is well-known that $gr^n_\gamma K_m(F)_\mathbb{Q} \neq 0$ only if $m = 2n-1$. So $H^i(L(F)_\bullet) = 0$ for $i \geq 2$ and hence $L(F)_\bullet$ is a free graded Lie algebra. Further, A. Borel proved ([Bo1-2], see also s.2 of §2) that for $m > 1$

$$\dim\, K_{2m-1}(F)_\mathbb{Q} = d_m := \begin{cases} r_1 + r_2, & \text{if } m \text{ is odd} \\ r_2, & \text{if } m \text{ is even} \end{cases} \tag{6}$$

So $L(F)_\bullet$ is generated by $(F^*_\mathbb{Q})^\vee$ in degree -1 and vector spaces of dimension d_m in degrees $-m = -2, -3, \ldots$.

Example 1.6 F is a finite field. Then $K_*(F)_\mathbb{Q} = 0$, (see [Q2]), so $L(F)_\bullet = 0$. This agrees with the fact that the category $\mathcal{M}_T(F)$ should be semisimple because Frobenius acts on simple objects $\mathbb{Q}(j)$ with different eigenvalues q^{-j}.

Let us denote by F_0 the subfield of constants in a field F (i.e. F_0 is the closure in F of the prime field).

Rigidity Conjecture 1.7 (A.A. Beilinson) *The canonical map* $K_*(F_0) \to K_*(F)$ *induces an isomorphism* $gr^n_\gamma K_{2n-1}(F_0) \xrightarrow{\sim} gr^n_\gamma K_{2n-1}(F)$.

Example 1.8 Now let char $F = p > 0$. Then example 1.6 together with the rigidity conjecture implies that $gr^n_\gamma K_{2n-1}(F_0)_\mathbb{Q}$ should be zero for $n \geq 2$. This means that $L(F)_\bullet$ is generated by $(F^*_\mathbb{Q})^\vee$ sitting in degree -1.

3. The structure of $L(F)_\bullet$. Set

$$I(F)_\bullet := \oplus_{n=2}^\infty L(F)_{-n}$$

Conjecture 1.9 *$I(F)_\bullet$ is a free graded Lie algebra.*

Our next aim is to construct explicitly the quotient $L_\bullet/[I_\bullet, I_\bullet]$. There is the following extension

$$0 \to I_\bullet/[I_\bullet, I_\bullet] \longrightarrow L_\bullet/[I_\bullet, I_\bullet] \longrightarrow L_\bullet/I_\bullet \longrightarrow 0 \ . \tag{7}$$

Let $\mathfrak{n}$ be a nilpotent Lie algebra. Then $H_1(\mathfrak{n}) = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ can be interpreted as a space of generators of $\mathfrak{n}$ (as a Lie algebra) and $H_2(\mathfrak{n})$ as a space of relations between generators; $\mathfrak{n}$ is free if and only if $H_2(\mathfrak{n}) = 0$. If $\mathfrak{n}$ is free then $H_i(\mathfrak{n}) = 0$ for $i \geq 2$.

Returning to (1.7) we see that the left space in (1.6) is just the space of generators of $I_\bullet$. So conjecture 1.9 together with explicit construction of extension (1.7) will give us in particular a complete description of the ideal $I_\bullet$. The quotient $L_\bullet/I_\bullet$ is abelian and as a $\mathbb{Q}$-vector space is isomorphic to $L_{-1}^\vee \cong (F_\mathbb{Q}^*)^\vee$ (see (1.5)). The including $L_{-1} \hookrightarrow L_\bullet$ provides canonical splitting $s : L_\bullet/I_\bullet \to L_\bullet/[I_\bullet, I_\bullet]$ of extension (1.7) as a $\mathbb{Q}$-vector spaces; the action of $L_\bullet$ on $I_\bullet$ gives the action of $L_\bullet/I_\bullet$ on $H_1(I_\bullet)$. Let $H_1^{(-n)}(I_\bullet)$ be the component of grading $-n$ of $H_1(I_\bullet)$. Then to construct $L_\bullet/[I_\bullet, I_\bullet]$ we need to define the following data:

i) A graded $\mathbb{Q}$-vector space $H_1(I_\bullet) = \oplus_{n=+2}^{\infty} H_1^{(-n)}(I_\bullet)$ (1.8a)

ii) A map $(F_\mathbb{Q}^*)^\vee \wedge (F_\mathbb{Q}^*)^\vee \to H_1^{(2)}(I_\bullet)$ (1.8b)
(this will be the commutator $[s(L_\bullet/I_\bullet), s(L_\bullet/I_\bullet)]$)

iii) Maps $(F_\mathbb{Q}^*)^\vee \otimes H_1^{(-(n-1))}(I_\bullet) \to H_1^{(-n)}(I_\bullet)$ (1.8c)

Dualising (1.8) we get

$$f_2 : H^1_{(2)}(I_\bullet) \to \wedge^2 F_\mathbb{Q}^* \tag{1.9a}$$

$$f_n : H^1_{(n)}(I_\bullet) \to H^1_{(n-1)}(I_\bullet) \otimes F_\mathbb{Q}^* \tag{1.9b}$$

This data will be defined in the next section.

4. The groups $\mathcal{R}_n(F)$. Let us define by induction subgroups $\mathcal{R}_n(F) \subset \mathbb{Z}[P_F^1]$, $n \geq 1$. Set

$$\mathcal{B}_n(F) := \mathbb{Z}[P_F^1]/\mathcal{R}_n(F)$$

The subgroup $\mathcal{R}_1(F)$ was already defined in such a way that $\mathcal{B}_1(F) = F^*$:

$$\mathcal{R}_1(F) := (\{x\} + \{y\} - \{xy\}, (x, y \in F^*); \{0\}; \{\infty\}) \ .$$

Consider homomorphisms

$$\mathbb{Z}[P^1_F] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F)\otimes F^* & : \quad n\geq 3 \\ \wedge^2 F^* & : \quad n=2 \end{cases}$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1}\otimes x & : \quad n\geq 3 \\ (1-x)\wedge x & : \quad n=2 \end{cases}$$

$$\delta_n : \{\infty\}, \{0\}, \{1\} \mapsto 0 \tag{10}$$

Here $\{x\}_n$ is the projection of $\{x\}$ in $\mathcal{B}_n(F)$. Set

$$\mathcal{A}_n(F) := \text{Ker } \delta_n \ .$$

Any element $\alpha(t) = \Sigma n_i\{f_i(t)\} \in \mathbb{Z}[P^1_{F(t)}]$ has a specialization $\alpha(t_0) := \Sigma n_i\{f_i(t_0)\} \in \mathbb{Z}[P^1_F]$, $t_0 \in P^1_F$. (It is correctly defined even if t_0 is a pole of $f_i(t)$, in this case $f_i(t_0) = \infty \in P^1_F$).

Definition 1.10 *$\mathcal{R}_n(F)$ is generated by elements $\alpha(0) - \alpha(1)$ where $\alpha(t)$ runs all elements of $\mathcal{A}_n(F(t))$, and also $\{\infty\}, \{0\}$.*

Lemma 1.11 $\delta_n(\mathcal{R}_n(F)) = 0$.

Proof. See proof of lemma 1.16 in [G2]. □

So we get

$$\delta : \mathcal{B}_n(F) \rightarrow \begin{cases} \mathcal{B}_{n-1}(F)\otimes F^* & : \quad n\geq 3 \\ \wedge^2 F^* & : \quad n=2 \end{cases}$$

Let me give some examples of elements of $\mathcal{R}_n(F)$.

Example 1.12 $\{x\} + \{x^{-1}\}$ and $\{x\} + \{1-x\} \in \mathcal{R}_2(F)$. Indeed, $\delta_2(\{x\} + \{x^{-1}\}) = (1-x)\wedge x + (1-x^{-1})\wedge x^{-1} = 0$ in $\wedge^2 F(t)^*$ modulo 2-torsion. On the other hand, $\{x\} + \{x^{-1}\}\,|_{x=\infty} \in \mathcal{R}_2(F)$ by definition. The same arguments work for $\{x\} + \{1-x\}$.

Example 1.13 $\{x\} + (-1)^n\{x^{-1}\} \in \mathcal{R}_n(F)$. Indeed, by induction $\delta_n(\{x\} + (-1)^n\{x^{-1}\}) = (\{x\} + (-1)^{n-1}\{x\})\otimes x \in \mathcal{R}_{n-1}(F(t))\otimes F(t)^*$ and $\{x\} + (-1)^n\{x^{-1}\}|_{x=\infty} \in \mathcal{R}_n(F)$ by definition. In particular, $2\cdot\{1\} \in \mathcal{R}_{2m}(F)$. (Put $x = 1, n = 2m$). We will prove in the next section that $\{1\} \notin \mathcal{R}_{m+1}(\mathbb{C})$ (see example 1.18).

5. Motivation: polylogarithms. The classical n-logorithm can be defined on the unit disk $|z| \leq 1$ by absolutely convergent series

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \ .$$

We have

$$\begin{aligned} Li_1(z) &= -\log(1-z) \\ d\ Li_n(z) &= Li_{n-1}(z)\ d\log z\ . \end{aligned} \tag{11}$$

So using the formula

$$Li_n(z) = \int_0^z Li_{n-1}(w)\frac{dw}{w}$$

we can continue analytically $Li_n(z)$ to a multivalued analytical function on $\mathbb{C}P^1\backslash\{0,1,\infty\}$. However n-logarithm has a remarkable single-valued version $(n \geq 2)$:

$$\begin{aligned} \mathcal{L}_n(z) &:= \begin{matrix} \text{Re} & (n: \text{ odd}) \\ \text{Im} & (n: \text{ even}) \end{matrix} \left(\sum_{k=0}^{n} \frac{B_k \cdot 2^k}{k!} \log^k |z| \cdot Li_{n-k}(z) \right)\ , \quad n \geq 2 \\ \mathcal{L}_1(z) &:= \log|z| \end{aligned}$$

Let me note that

$$\mathcal{L}_2(z) = \text{Im}\ (Li_2(z)) + \arg(1-z)\log|z| \tag{12}$$

is the well-known Bloch-Wigner function, and

$$\mathcal{L}_3(z) = \text{Re}(Li_3(z) - \log|z| \cdot Li_2(z) - \frac{1}{3}\log^2|z|\log(1-z))$$

was used in [G1]. The functions $\mathcal{L}_n(z)$ for arbitrary n were written by D. Zagier [Z1], who proved the following theorem:

Theorem 1.14 $\mathcal{L}_n(z)$ is continuous on $\mathbb{C}P^1$ for $n \geq 2$.

It is clear that then $\mathcal{L}_n(z)$ is real-analytical on $\mathbb{C}P^1\backslash\{0,1,\infty\}$.

The Hodge-theoretical interpretation of these functions was given by A.A. Beilinson and P. Deligne (see, for example, [D2]).

Any real-valued function, and in particular $\mathcal{L}_n(z)$, defines a homomorphism

$$\begin{aligned} \tilde{\mathcal{L}}_n : \mathbb{Z}[P^1_{\mathbb{C}}] &\longrightarrow \mathbb{R} \\ \{z\} &\mapsto \mathcal{L}_n(z)\ . \end{aligned}$$

Theorem-motivation 1.15 $\tilde{\mathcal{L}}_n(\mathcal{R}_n(\mathbb{C})) = 0$

Proof. Let us prove the theorem in the case $n = 2$ for beginning.

Lemma 1.16 *Let* $\alpha(t) = \Sigma n_i\{f_i(t)\} \in \mathbb{Z}[P^1_{\mathbb{C}(t)}]$. *If*

$$\delta_2\alpha(t) := \Sigma n_i(1 - f_i(t)) \wedge f_i(t) = 0$$

in $\wedge^2\mathbb{C}(t)^*$ then $d(\Sigma n_i\mathcal{L}_2(f_i(z))) = 0$.

It follows immediately from the lemma that $\tilde{\mathcal{L}}_2(\alpha(0) - \alpha(1)) = 0$ and so $\tilde{\mathcal{L}}_2(\mathcal{R}_2(\mathbb{C})) = 0$.

Proof of Lemma 1.16 Let us consider the following diagram

$$\begin{array}{ccc} \mathbb{Z}[P^1_{\mathbb{C}(t)}] & \xrightarrow{\delta_2} & \wedge^2\mathbb{C}(t)^* \\ \tilde{\mathcal{L}}_2 \downarrow & & \downarrow r_2 \\ S^0(\mathbb{C}P^1) & \xrightarrow{d} & S^1(\mathbb{C}P^1) \end{array} \qquad (1.13)$$

$$r_2(f \wedge g) := -\log|f|d\arg g + \log|g|d\arg f\ .$$

Here $S^i(\mathbb{C}P^1)$ is the space of smooth i-forms each defined on an appropriate Zariski open domain of $\mathbb{C}P^1$ ($= C^\infty$ i-forms at the generic point of $\mathbb{C}P^1$).

The formula

$$d\mathcal{L}_2(z) = -\log|1 - z|d\arg z + \log|z|d\arg(1 - z)$$

provides the commutativity of the diagram (1.13). So if $\alpha(t) \in \mathcal{A}_2(\mathbb{C}(t))$, then

$$0 = r_2 \circ \delta_2(\alpha(t)) = d \circ \tilde{\mathcal{L}}_2(\alpha(t)) \stackrel{\text{def}}{=} d(\Sigma n_i\mathcal{L}_2(f_i(z)))\ .$$

Set

$$\widehat{\mathcal{L}_n}(z) = \begin{cases} \mathcal{L}_n(z) & n: \text{ odd} \\ i\mathcal{L}_n(z) & n: \text{ even} \end{cases}$$

Then we have for $n \geq 3$

$$d\widehat{\mathcal{L}_n}(z) = \widehat{\mathcal{L}_{n-1}}(z)d(i\arg z) - \sum_{k=2}^{n-2}\frac{B_k \cdot 2^k}{k!}\log^{k-1}|z| \cdot \widehat{\mathcal{L}_{n-k}}(z) \cdot d\log|z|$$
$$-\frac{B_n \cdot 2^n}{n!}\log^{n-1}|z|(\log|z|d\log|1 - z| - \log|1 - z|d\log|z|)\ . \qquad (14)$$

It is interesting that in this formula the same coefficients appear as in (1.12).

The proof of the theorem in the case $n \geq 3$ is based on this formula and the following commutative diagram it provides

$$\begin{array}{ccc} \mathbb{Z}[P^1_{\mathbb{C}(t)}] & \longrightarrow & B_{n-1}(\mathbb{C}(t)) \otimes \mathbb{C}(t)^* \\ \Big\downarrow {\scriptstyle \widetilde{\hat{\mathcal{L}}_n}} & & \Big\downarrow {\scriptstyle r_n} \\ S^0(\mathbb{C}P^1) & \xrightarrow{\quad d \quad} & S^1(\mathbb{C}P^1) \end{array}$$

where

$$\begin{aligned} r_n(\{f(t)\}_{n-1} \otimes g(t)) &:= \hat{\mathcal{L}_{n-1}}(f(t)) di \arg g(t) - \\ &- \sum_{k=2}^{n-2} \frac{B_k \cdot 2^k}{k!} \log^{k-1}|f(t)| \cdot \hat{\mathcal{L}}_{n-k}(f(t)) d \log |g(t)| - \\ &- \frac{B_n \cdot 2^n}{n!} \log |g(t)| \cdot \log^{n-3}|f(t)| \cdot (\log |f(t)| d \log |1 - f(t)| - \\ &\quad - \log |1 - f(t))| d \log |f(t)|) \end{aligned}$$

There are 3 terms in this formula. Each of them is a homomorphism from $B_{n-1}(\mathbb{C}(t)) \otimes \mathbb{C}(t)^*$ to $S^1(\mathbb{C}P^1)$: the first by induciton; the second because it is a composition of the homomorphism

$$\begin{array}{ccc} B_{n-1}(\mathbb{C}(t)) \otimes \mathbb{C}(t)^* & \longrightarrow & B_{n-k}(\mathbb{C}(t)) \otimes S^{k-1}\mathbb{C}(t)^* \otimes \mathbb{C}(t)^* \\ {\scriptstyle \delta(k-1) \otimes id} \searrow & & \nearrow {\scriptstyle id \otimes \text{ projection } \otimes id} \\ & B_{n-k}(\mathbb{C}(t)) \otimes \underbrace{\mathbb{C}(t)^* \otimes \ldots \otimes \mathbb{C}(t)^*}_{k \text{ times}} & \end{array}$$

($\delta(1) := \delta$ and $\delta(k) := (\delta \otimes id) \circ \delta(k-1)$) with the obvious homomorphism from $B_{n-k}(\mathbb{C}(t)) \otimes S^{k-1}\mathbb{C}(t)^* \otimes \mathbb{C}(t)$ to $S^1(\mathbb{C}P^1)$; and finally the third one is the composition of the homomorphism

$$\begin{array}{ccc} B_{n-1}(\mathbb{C}(t)) \otimes \mathbb{C}(t)^* & \longrightarrow & \wedge^2\mathbb{C}(t)^* \otimes S^{n-2}\mathbb{C}(t)^* \\ {\scriptstyle \delta(n-3) \otimes id} \searrow & & \nearrow {\scriptstyle \delta \otimes \text{ projection}} \\ & B_2(\mathbb{C}(t)) \otimes \mathbb{C}(t)^* \otimes \ldots \otimes \mathbb{C}(t)^* & \end{array}$$

with $r_2 \otimes \sqcap \log|\cdot|$.

For another formula for $d\mathcal{L}_n(z)$ (without Bernoulli numbers on the right-hand side) see [Z1], where D. Zagier suggests a slightly different definition of the "subgroup of functional equations" for $\mathcal{L}_n(z)$.

Theorem 1.17 *Suppose that for some $f_i(t) \in \mathbb{C}(t)$ $\sum_i a_i \mathcal{L}_n(f_i(t)) = 0$. Then*

$$\sum_i a_i \left(\{f_i(t)\} - \{f_i(0)\}\right) \in R_n(\mathbb{C}) \ .$$

See proposition 4.9 for the case $n = 2$. The proof in the general case follows the same idea: to study singularities of $d(\sum a_i \mathcal{L}_n(f_i(t))$ using formula (1.14) $\square$

Theorem 1.5 permits us to prove that the quotient $\mathcal{A}_n(F)/\mathcal{R}_n(F)$ can be nontrivial. The simplest example is:

Example 1.18 $\{1\} \notin \mathcal{R}_{2n+1}(\mathbb{C})$ because $\mathcal{L}_{2n+1}(1) = \zeta_{\mathbb{Q}}(2n+1) \neq 0$. (Compare with example 1.13 where we proved that $2 \cdot \{1\} \in \mathcal{R}_{2n}(F)$).)

Remark Let us denote by $F(X)$ the field of rational functions on a curve X/F. The proof of theorem 1.15 suggest

Definition 1.19 *$\mathcal{R}'_n(F)$ is generated by elements $\alpha(t_0) - \alpha(t_1)$ where t_0, t_1 runs all F-points of X, X runs all curves over F and $\alpha(t)$ runs all elements of $\mathcal{A}_n(F(X))$.*

The previous definition uses only P^1 instead of all curves over F. However, I believe that the natural map $\mathcal{R}_n(F) \to \mathcal{R}'_n(F)$ is an isomorphism. In fact this is equivalent to the rigidity conjecture 1.7 (see s. 9 of §1 in [G2]).

6. The main conjecture. Now we are ready to formulate the conjecture about the structure of the Lie algebra $L(F)_\bullet$. As was explained in s. 3 to describe the ideal $I_\bullet$ and extension

$$\begin{array}{ccccc} 0 \to H_1(I) \to L_\bullet/[I_\bullet, I_\bullet] \to & L_\bullet/I_\bullet & \to 0 \\ & \| & \\ & (F^*_{\mathbb{Q}})^\vee & \end{array}$$

is sufficient to define the following data (see (1.9)).

$$\begin{array}{ll} \text{i)} & H^1(I_\bullet) = \oplus_{n=2}^{\infty} H^1_{(n)}(I_\bullet) \\ \text{ii)} & f_2 : H^1_{(2)}(I_\bullet) \to \wedge^2 F^*_{\mathbb{Q}} \\ \text{iii)} & f_n : H^1_{(n)}(I_\bullet) \to H^1_{(n-1)}(I_\bullet) \otimes F^*_{\mathbb{Q}} \end{array} \qquad (15)$$

Conjecture 1.20 *For an arbitrary field F*

a) *$I(F)_\bullet$ is a free graded pro-Lie algebra*

b) $H^1_{(n)}(I(F)_\bullet) \cong \mathcal{B}_n(F)_\mathbb{Q} \qquad n \geq 2$, i.e. *$I(F)_\bullet$ is generated as a graded Lie algebra by the spaces $\mathcal{B}_n(F)^\vee$ seating in degree $-n$.*

c) *$L_\bullet / I_\bullet \cong (F^*_\mathbb{Q})^\vee$ and f_n coincides with*

$$\delta : \mathcal{B}_{n-1}(F)_\mathbb{Q} \to \begin{cases} \mathcal{B}_n(F)_\mathbb{Q} \otimes F^*_\mathbb{Q} & : n \geq 3 \\ \wedge^2 F^*_\mathbb{Q} & : n = 2 \end{cases}$$

2 Corollaries

1. A candidate for the Beilinson-Lichtenbaum complexes. Let us compute $H^*_{(n)}(L(F)_\bullet)$ using the Hochshild-Serre spectral sequence for the ideal $I_\bullet$ and conjecture 1.20. We get

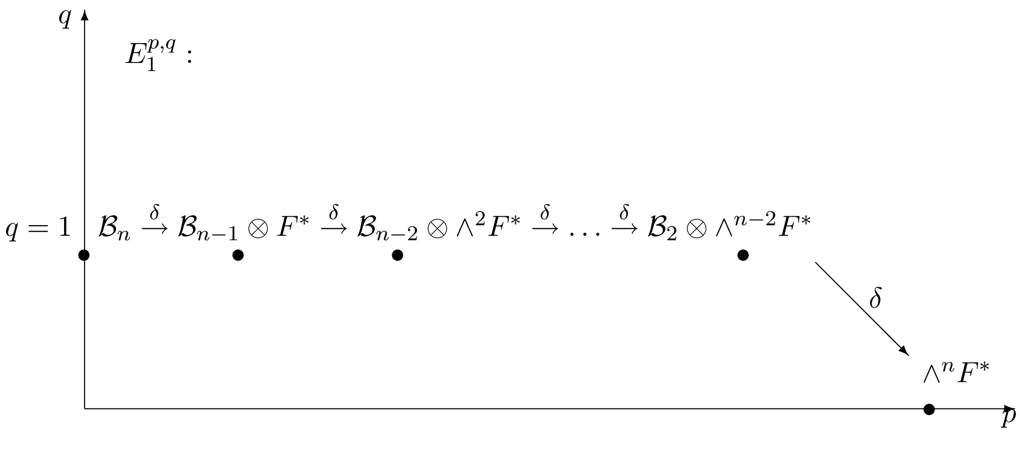
$$E_1^{p,q} = C^p(L_\bullet/I_\bullet, H^q_{(n-p)}(I_\bullet)) = \begin{cases} \wedge^p F_\mathbb{Q} \otimes \mathcal{B}_{n-p}(F)_\mathbb{Q} : & q = 1 \\ \wedge^n F^*_\mathbb{Q} & : q = 0, n = p \\ 0 & : \text{otherwise} \end{cases}$$

The action of $L_\bullet/I_\bullet$ on $\oplus_{m=2}^\infty H_1^{(-m)}(I_\bullet)$ is given by maps f^*_m dual to f_m $(m \geq 3)$. So the differential

$$d_1^{p,1} : \mathcal{B}_{n-p}(F)_\mathbb{Q} \otimes \wedge^p F^*_\mathbb{Q} \to \mathcal{B}_{n-p-1}(F)_\mathbb{Q} \otimes \wedge^{p+1} F^*_\mathbb{Q}$$

is given by the formula $(n - p \geq 3)$

$$\delta : \{x\}_{n-p} \otimes y_1 \wedge \ldots \wedge y_p \mapsto \{x\}_{n-p-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_p$$



The only non-trivial higher differential is

$$\begin{aligned} d_2^{n-2,1} \quad : \quad & \mathcal{B}_2(F)_{\mathbb{Q}} \otimes \wedge^{n-2} F^*_{\mathbb{Q}} \to \wedge^n F^*_{\mathbb{Q}} \\ & \{x\}_2 \otimes y_1 \wedge \ldots y_{n-2} \mapsto (1-x) \wedge x \wedge y_1 \ldots \wedge y_{n-2} \ . \end{aligned}$$

So we get the following complex $\Gamma(F,n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \wedge^2 F^* \xrightarrow{\delta} \mathcal{B}_2 \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

where $\mathcal{B}_n \equiv \mathcal{B}_n(F)$ placed in degree 1 and

$$\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \to \delta(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_1$$

has degree +1. Conjecture 1.19 together with (1.3) implies

Conjecture 2.1 $H^i(\Gamma(F,n)_{\mathbb{Q}}) \cong gr^n_\gamma K_{2n-i}(F)_{\mathbb{Q}}$
This conjecture gives a symbolic description of K-groups.

Example Let $F = \mathbb{Q}$. We showed in example 1.17 that $\{1\} \notin \mathcal{R}_{2n+1}(\mathbb{Q})$ for $n \geq 1$. So $\{1\}$ should represent a non-trivial element in $gr^{2n+1}_\gamma K_{4n+1}(\mathbb{Q})$. Note that $\dim K_m(\mathbb{Q}) = \begin{cases} 1 & \text{for } m = 4n+1 \\ 0 & \text{otherwise} \end{cases}$

Complexes $\Gamma(F,n)_{\mathbb{Q}}$ should satisfy Beilinson- Lichtenbaum axioms.

In fact conjecture 2.1 is equivalent to conjecture 1.19 if we assume (1.3). More precisely, let us suppose that there exist homomorphisms $\psi_n : \mathcal{B}_n(F)_{\mathbb{Q}} \to L(F)^\vee_{-n}$ such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{B}_2(F)_{\mathbb{Q}} & \xrightarrow{\delta} & \wedge^2 F^*_{\mathbb{Q}} \\ \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \wedge^2\psi_1} \\ L(F)^\vee_{-2} & \xrightarrow{\partial} & \wedge^2 L(F)^\vee_{-1} \end{array} \qquad (2.1a)$$

$$\begin{array}{ccc} \mathcal{B}_n(F)_{\mathbb{Q}} & \xrightarrow{\delta} & \mathcal{B}_{n-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}} \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_{n-1}\otimes\psi_1} \\ L(F)^\vee_{-n} & \xrightarrow{\partial_{(1)}} & L(F)^\vee_{-(n-1)} \otimes L(F)^\vee_{-1} \end{array} \qquad (2.1b)$$

Here $\partial_{(1)}$ is the $L^{\vee}_{-(n-1)} \otimes L^{\vee}_{-1}$-component of Δ. Then we get a homomorphism of complexes

$$\Psi_n : \Gamma(F,n)_{\mathbb{Q}} \to \wedge^{\bullet}_{(n)}(L(F)^{\vee}_{\bullet}) \; . \tag{2}$$

Theorem 2.2 *Suppose that there exists a graded Lie algebra* $L_{\bullet} = \oplus_{n=1}^{\infty} L_{-n}$ *and homomorphisms* $\psi_n : \mathcal{B}_n \to L^{\vee}_{-n}$ *such that diagrams 2.1a), 2.1b) are commutative and* Ψ_n *is a quasiisomorphism for* $n \geq 1$*. Then*

a) $I_{\bullet} := \oplus_{n=2}^{\infty} L_{-n}$ *is a graded Lie algebra*

b) $\psi_n : \mathcal{B}_n \to H^1_{(n)}(I_{\bullet})$ *is an isomorphism for any* $n \geq 2$

c) *Maps* f_n *describing the quotient* $L_{\bullet}/[I_{\bullet}, I_{\bullet}]$ *(see (1.15)) coincides with*

$$\delta : \mathcal{B}_n \to \begin{cases} \mathcal{B}_{n-1} \otimes F^*_{\mathbb{Q}} & : n \geq 3 \\ \wedge^2 F^*_{\mathbb{Q}} & : n = 2 \end{cases}$$

For the proof of the theorem, see proof of proposition 1.26 in [G2] □

Our next purpose will be to show that conjecture 2.1 in the case when F is a number field implies Zagier's conjecture about the values of Dedekind zeta functions $\zeta_F(n)$. But first of all we need to recall the Borel theorems.

2. The Borel theorems. Set $R(n) = (2\pi i)^n R \subset \mathbb{C}$ and $X_F := \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$. Let us define the Borel regulator $r_m : K_{2m-1}(F) \to X_F \otimes R(m-1)$. The Hurewicz map gives a canonical homomorphism

$$\begin{aligned} K_{2m-1}(F) := \pi_{2m-1}(BGL(F)^+) \to H_{2m-1}(BGL(F)^+, \mathbb{Z}) = \\ = H_{2m-1}(GL(F), \mathbb{Z}) \; . \end{aligned} \tag{3}$$

For every embedding $\sigma : F \hookrightarrow \mathbb{C}$ we have a homomorphism

$$H_{2m-1}(GL(F), \mathbb{Z}) \to H_{2m-1}(GL(\mathbb{C}), \mathbb{Z}) \; . \tag{4}$$

There is a canonical pairing

$$H^{2m-1}(GL(\mathbb{C}), R(m-1)) \times H_{2m-1}(GL(\mathbb{C}), \mathbb{Z}) \xrightarrow{<,>} R(m-1) \; . \tag{5}$$

Let us define a canonical element

$$b_{2m-1} \in H^{2m-1}_{cts}(GL(\mathbb{C}), R(m-1)) \subset H^{2m-1}(GL(\mathbb{C}), R(m-1)) \; .$$

Recall that (cf. [Bo1]) $H^*_{cts}(GL(\mathbb{C}), R) \cong H^*_{\text{top}}(U, R)$ where $H^*_{\text{top}}(U, R)$ is the cohomology of the infinite unitary group, considered as a topological space. Further,

$$H^*_{\text{top}}(U, \mathbb{Z}) = H^*(S^1 \times S^3 \times S^5 \times \ldots, \mathbb{Z}) = \wedge^*_{\mathbb{Z}}(u_1, u_3, \ldots)$$

where u_i denotes the class of the sphere S^i.

Combining the above isomorphisms we get an isomorphism

$$\varphi : H^*_{\text{cts}}(GL(\mathbb{C}), R) \xrightarrow{\sim} \wedge^*_{\mathbb{Z}}(u_1, u_3, \ldots) \otimes R\ . \tag{6}$$

Set $b'_{m-1} := 2\pi \cdot \varphi^{-1}(u_{2m-1})$ and

$$b_{2m-1} := (2\pi i)^{m-1} \cdot b'_{2m-1} \in H^*_{\text{cts}}(GL(\mathbb{C}), R(m-1))\ .$$

So combining this with (2.3)–(2.5) we get

$$K_{2m-1}(F) \longrightarrow \oplus_{\text{Hom}(F,\mathbb{C})} K_{2m-1}(\mathbb{C}) \longrightarrow X_F \otimes R(m-1)\ .$$

It is known that if $\lambda \in H^d_{\text{cont}}(GL(\mathbb{C}), R)$ and c^* denotes the involution defined by complex conjugation c, then in (2.6) $c^*\varphi(\lambda) = (-1)^d \varphi(c^*\lambda)$, where c acts also on $S^{2m-1} \subset \mathbb{C}^m$. Note that $c^* u_{2m-1} = (-1)^m u_{2m-1}$. So we see that

$$r_m : K_{2m-1}(F) \longrightarrow [X_F \otimes R(m-1)]^+ = R^{d_m}$$

where on the right-hand side stands the subspace of invariants of the action of c and

$$d_m = \begin{cases} r_1 + r_2, & \text{if } m \text{ is odd} \\ r_2, & \text{if } m \text{ is even} \end{cases}$$

is its dimension. Here r_1 resp. r_2 the number of real resp. complex places, so $[F : Q] = r_1 + 2r_2$.

In fact, we construct a homomorphism

$$r_m^{(n)} : \text{Prim } H_{2m-1}(GL_n(F), \mathbb{Z}) \to [X_F \otimes R(m-1)]^+\ .$$

For any lattice Λ of $(X_F \otimes R(m-1))^+$ define its (co)volume vol Λ by

$$\det(\Lambda) = \text{vol}(\Lambda) \cdot \det[X_F \otimes R(m-1)]^+\ .$$

Theorem 2.3 (Borel [Bo1], [Bo2]). *For every $m \geq 2$ and sufficiently large n*

a) Im $r_m^{(n)}$ *is a lattice in* $(X_F \otimes R(m-1))^+$.

b) $R_m := \text{vol}(\text{Im } r_m^{(n)}) \sim \mathbb{Q}^* \cdot \lim_{s \to 1-m}(s-1+m)^{-d_m}\zeta_F(s)$.

Here $a \sim \mathbb{Q}^* b$ means that $a = \kappa b$ for some $\kappa \in \mathbb{Q}^*$.

Remark 2.4 The functional equation for $\zeta_F(s)$ shows that

$$\zeta_F(m) \sim \mathbb{Q}^* \cdot \pi^{(r_1+2r_2-d_m)\cdot m} \cdot |d_F|^{-\frac{1}{2}} \cdot R_m$$

where d_F is the discriminant of F.

3. Zagier's conjecture. According to conjecture 2.1 we have an isomorphism

$$H^1(\Gamma(\mathbb{C}, n)_{\mathbb{Q}}) \equiv \text{Ker}(\mathcal{B}_n(\mathbb{C})_{\mathbb{Q}} \xrightarrow{\delta} \mathcal{B}_{n-1}(\mathbb{C})_{\mathbb{Q}} \otimes \mathbb{C}^*) \cong gr^n_\gamma K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \ .$$

Recall that there is a homomorphism $\mathcal{L}_n : \mathcal{B}_n(\mathbb{C}) \to R$. We expect that the restriction of this homomorphism to the subgroup $H^1(\Gamma(\mathbb{C}, (n)_{\mathbb{Q}}) \subset \mathcal{B}_n(\mathbb{C})_{\mathbb{Q}}$ coincides with the Borel regulator (the reasons can be found in §1 of [G2]). So applying the Borel theorem we come to the following conjecture.

Conjecture 2.5 *Let F be a number field and σ_j the set of all possible embeddings $F \hookrightarrow \mathbb{C}, (1 \leq j \leq r_1 + 2r_2)$ numbered so that $\sigma_{r_1+k} = \overline{\sigma_{r_1+r_1+k}}$. Then there exists elements*

$$y_1, \ldots y_{d_m} \in \text{Ker}(\mathcal{B}_n(F)_{\mathbb{Q}} \xrightarrow{\delta} \mathcal{B}_{n-1}(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}})$$

such that

$$\zeta_F(n) = \pi^{(r_1+2r_2-d_n)\cdot n}|d_F|^{-\frac{1}{2}}\det|\mathcal{L}_n(\sigma_j(y_i))| \ , \quad (1 \leq i, j \leq d_n) \ .$$

This conjecture was stated by Don Zagier, who proved it for $s = 2$ [Z2] and using a computer gave an impressive list of numerical examples (see [Z1]). The case $s = 2$ follows also from the Borel theorem and the results of S. Bloch [Bl1] and A. Suslin [S1]. A complete proof for the case $s = 3$ will be given in §3 (see also [G1] and [G2]).

4. A topological consequence of conjecture 1.9. We will show that in the Beilinson's World (*a world where his conjectures are theorems*) conjecture 1.9 implies that commutant of the maximal Tate quotient of the pronilpotent completion of the classical fundamental group of the generic point of an arbitrary complex variety over $\mathbb{C}$ should be free graded pro-Lie algebra.

Recall that A.A. Beilinson conjectured ([B1]) that for arbitrary scheme X there exists a mixed Tate category $\mathcal{M}_T(X)$ of mixed motivic Tate sheaves over X. In the special case $X = \text{Spec } F$, F is a field, $\mathcal{M}_T(\text{Spec } F)$ is just the category $\mathcal{M}_T(F)$ discussed in s. 1–2 of §1. Let us denote by $L(X)_\bullet$ the corresponding mixed Tate Lie algebra. Any morphism of schemes $f : X \to Y$ defines a Tate functor $f^* : \mathcal{M}_T(Y) \to \mathcal{M}_T(X)$ ("inverse image" of mixed Tate sheaves) such that $\omega_{\mathcal{M}_T(X)} f^* = \omega_{\mathcal{M}_T(Y)}$ ($\omega_{\mathcal{M}}$ is the canonical fiber functor for a mixed Tate category $\mathcal{M}$). So we have a morphism $f_* : L(X)_\bullet \to L(Y)_\bullet$ of the corresponding mixed Tate Lie algebras. In particular, if X is a scheme over field F, we have the map $p_* : L(X)_\bullet \to L(\text{Spec } F)_\bullet$ that should be surjective because p^* is fully faithful. Put $L(X)^g_\bullet := \text{Ker } p_*$ (the "geometrical part of $L(X)_\bullet$"). We get the following exact sequence

$$0 \to L(X)^g_\bullet \to L(X)_\bullet \xrightarrow{p_*} L(\text{Spec } F)_\bullet \to 0 \ .$$

Note that

$$[L(X)^g_\bullet, L(X)^g_\bullet] \subset L(X)^g_{\leq -2} \ .$$

(It was proved in [B2] (see lemma 1.2.1) that $L(X)^g_\bullet$ is generated by $L(X)^g_{-1}$, so $[L(X)^g_\bullet, L(X)^g_\bullet] = L(X)^g_{\leq -2}$, but we will not use this fact).

Let $\eta = \text{Spec } k(X)$ be the generic point of X. Then according to conjecture 1.9 the (graded) Lie algebra $L(\eta)_{\leq -2}$ is free. Therefore its subalgebra $[L(\eta)^g_\bullet, L(\eta)^g_\bullet]$ is also free.

Now let X be a smooth algebraic variety over $\mathbb{C}$. I need to explain what is the *maximal Tate quotient of the pronilpotent completion of* $\pi_1(\text{Spec}\mathbb{C}(X))$. In [H-Z] R. Hain and S. Zucker defined category $\mathcal{H}^{\text{un}}_X$ of good unipotent variations of mixed R- Hodge structures over X ("good" means some growth conditions at infinity).

Fix any $x \in X$. Let $V \in Ob\mathcal{H}^{un}_x$ and V_x is the fiber of the local system underlying V at point x. Then the monodromy representation $\rho : \pi_1(X, x) \to \text{Aut}(V_x)$ is unipotent and hence defines an algebra homomorphism $\bar{\rho} : \mathbb{C}\pi_1(X, x)^\wedge \to \text{Aut}(V_x)$, where $\mathbb{C}\pi_1(X, x)^\wedge := \lim\limits_{\leftarrow} \mathbb{C}[\pi_1(X, x)]/J^r$, ($J$ is the kernel of the usual augmentation homomorphism). It is well-known that $\mathbb{C}\pi_1(X, x)^\wedge$ is a Hopf algebra in the category $\mathcal{H}$ of mixed R-Hodge structures and $\bar{\rho}$ is a mixed Hodge theoretic representation (i.e. representation in the category $\mathcal{H}$). R. Hain and S. Zucker proved the following theorem.

Theorem 2.6 *The monodromy representation functor $V \in \mathcal{H}^{un}_X \mapsto V_x$ defines an equivalence of categories*

$$\mathcal{H}^{un}_X \to \left\{ \begin{array}{c} \textit{category of mixed Hodge theoretic} \\ \textit{representations of } \mathbb{C}\pi_1(X, x)^\wedge \end{array} \right\}$$

The vector space underlying a Hodge structure $H \in \mathcal{H}$ is a fiber functor on the category $\mathcal{H}$. Composition of the functor $s_x : \mathcal{H}^{un}_X \to \mathcal{H}$, $s_x : V \mapsto$

V_x with this fiber functor gives a fiber functor on $\mathcal{H}_x^{un}$. Let us denote by $L(\mathcal{H})$ and $L(\mathcal{H}_x^{un}, x)$ the corresponding fundamental Lie algebras. We get an imbedding $s_x : L(\mathcal{H}) \to L(\mathcal{H}_x^{un})$. There is a canonical functor $c : \mathcal{H} \to \mathcal{H}_X^{un}$, $c(H)$ is a constant variation of the mixed Hodge structure H over X. So we get an epimorphism $c : L(\mathcal{H}_X^{un}, x) \to L(\mathcal{H})$. It is clear that $c \circ s_x = \text{id}$. Set $L(\mathcal{H}_X^{un}, x)^g := \text{Ker } c$. We get the following split exact sequence

$$0 \to L(\mathcal{H}_X^{un}, x)^g \to L(\mathcal{H}_X^{un}, x) \overset{s_x}{\underset{c}{\leftrightarrows}} L(\mathcal{H}) \to 0$$

Note that $s_x(L(\mathcal{H}))$ acts on the ideal $L(\mathcal{H}_X^{un}, x)^g$, and hence $L(\mathcal{H}_X^{un}, x)^g$ is equipped with canonical mixed Hodge structure. Further an $L(\mathcal{H}_X^{un}, x)$-module is just a mixed Hodge theoretic representation of $L(\mathcal{H}_X^{um}, x)^g$.

We have $\mathbb{C}\pi_1(X, x)^\wedge = \mathbb{C} \oplus \hat{J}$. The set of primitive elements

$$\mathfrak{G}_x := \{v \in \hat{J} : \Delta(v) = v\hat{\otimes}1 + 1\hat{\otimes}v\}$$

is a Lie algebra (Δ is the coproduct). The forgetting functor $\mathcal{H}_X^{un} \to \{\text{local systems on } X\}$ provides a homomorphism of Lie algebras $f_x : \mathfrak{G}_x \to L(\mathcal{H}_X^{un}, x)$ such that $c \circ f_x = 0$. So $f_x : \mathfrak{G}_x \to L(\mathcal{H}_X^{un}, x)^g$. Mixed Hodge structures $\mathfrak{G}_x$ form a good variation of mixed Hodge structures over X. So f_x is a morphism of mixed Hodge structures. Now it follows from theorem 2.6 that $f_x : \mathfrak{G}_x \xrightarrow{\sim} L(\mathcal{H}_X^{un}, , x)$ is an isomorphism.

Let $\mathcal{H}_X^T \subset \mathcal{H}_X^{un}$ be a subcategory of variations of mixed Hodge-Tate structures (i.e. $gr_{2n-1}^W V_x = 0$, $gr_{2n}^W V_x$ is a Hodge structure of type (n, n)). Then $L(\mathcal{H}_X^T, x)^g$ is maximal Tate quotient of $L(\mathcal{H}_X^{un}, x)^g$. If $\mathfrak{G}_x^T(X)$ is maximal Tate quotient of $\mathfrak{G}_x$, $\mathfrak{G}_x^T(X) \xrightarrow{\sim} L(\mathcal{H}_X^T, x)^g$ is an isomorphism. There is another fiber functor on category $\mathcal{H}_X^T$ that does not involve choice of $x \in X$: $H \in \mathcal{H}_X^T \mapsto \oplus_n gr_{2n}^W H$. Let us denote the corresponding geometrical Lie algebra $L(\mathcal{H}_X^T)^g$. Of course, $L(\mathcal{H}_X^T)^g \cong \mathfrak{G}_x^T(X)$. Set

$$L(\mathcal{H}_\eta^T)^g := \lim_{\overleftarrow{U \subset X}} L(\mathcal{H}_U^T)^g \ .$$

This is the **definition** of maximal Tate quotient of pronilpotent completion of fundamental group of generic point of a complex algebraic variety.

Conjecture 2.7 *The commutant of the Lie algebra $L(\mathcal{H}_\eta^T)^g$ is free.*

The Hodge-realization functor $\mathcal{M}_T(X) \to \mathcal{H}_X^T$ induces morphism $L(\mathcal{H}_X^T) \to L(\mathcal{M}_T(X))$ that should be isomorphism. (This follows from Beilinson's definition of mixed Hodge structure on $\mathbb{C}\pi_1(X, x)^\wedge$ and standard conjectures including the Hodge one - see [B2] and [B-D]). Therefore conjecture 2.7 is a corollary of conjecture 1.9 in the Beilinson's World.

It is interesting to compare conjecture 2.7 with the following one stated by F.A. Bogomolov.

Conjecture 2.8. Let Gal K be the maximal pro-p-quotient of the Galois group of the field K containing a nontrivial closed subfield. Then commutant [Gal K, Gal K] is free as a pro-p-group.

It is also reminiscent of the following Shafarevich's conjecture

Conjecture 2.9 [Gal $\bar{\mathbb{Q}}/\mathbb{Q}$, Gal $\bar{\mathbb{Q}}/\mathbb{Q}$] is free as a profinite group.

3 A proof of Zagier's conjecture about $\zeta_F(3)$

1. The Grassmanian complex ([S1], see also [BMS]). We will say that an m-tuple of vectors in n-dimensional vector space V^n is in a generic position if any $k \leq n$ vectors are linearly independent. **Configurations** of m vectors in V^n are n-tuples of vectors considered modulo $GL(V^n)$-equivalence. Let us denote by $\tilde{C}_m(n)$ the free abelian group generated by m-tuples of vectors in V^n in generic position. Let $C_m(n) := \tilde{C}_m(n)_{GL(V^n)}$ is coinvariants of the natural action of $GL(V^n)$ on $\tilde{C}_m(n)$. Then $C_m(n)$ is a free $GL(V^n)$-abelian group generated by configurations of m vectors in generic position in V^n. There is a differential

$$d : \tilde{C}_m(n) \to \tilde{C}_{m-1}(n); \quad d : (l_1, \ldots, l_m) \mapsto \sum_{i=1}^{m} (-1)^{i-1} (l_1, \ldots, \hat{l}_i, \ldots, l_m) .$$

We get a complex $(\tilde{C}_*(n), d)$ where $\tilde{C}_m(n)$ placed in degree $m-1$.

Lemma 3.1 $H_i(\tilde{C}_*(n)) = \begin{cases} 0 & \text{for } i \geq 1 \\ \mathbb{Z} & \text{for } i = 0 \end{cases}$ *if F is an infinite field.*

Proof. If $d(\Sigma n_j(l_1^{(j)}, \ldots, l_m^{(j)})) = 0$ choose a vector v in a generic position with respect to all $l_k^{(j)}$. Then $d(\Sigma n_j(v, l_1^{(j)}, \ldots, l_m^{(j)})) = \Sigma n_j(l_1^{(j)}, \ldots, l_m^{(j)})$ $\Box$

So $\tilde{C}_*(n)$ is a resolution of $\mathbb{Z}$, and therefore we have a map

$$H_i(GL_n(F)) \longrightarrow H_i(C_*(n)) . \tag{1}$$

2. Our strategy. We will work modulo 6-torsion. In the next section we will construct a homomorphism of complexes

$$
\begin{array}{ccccccc}
C_7(3) & \xrightarrow{d} & C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \\
 & & \downarrow{\scriptstyle f_6(3)} & & \downarrow{\scriptstyle f_5(3)} & & \downarrow{\scriptstyle f_4(3)} \\
0 & \longrightarrow & \mathcal{B}_3(F) & \xrightarrow{\delta} & \mathcal{B}_2(F) \otimes F^* & \xrightarrow{\delta} & \wedge^3 F^*
\end{array}
\qquad (3.2)
$$

and hence get a map

$$c_i(3) : H_i(GL_3(F)) \to H^{6-i}(\Gamma(F,3)) \ , \quad i = 3,4,5 \ .$$

Then we will construct a map $c_i(N) : H_i(GL_N(F)) \to H^{6-i}(\Gamma(F,3))$ such that the following diagram is commutative

$$
\begin{array}{ccc}
H_i(GL_3(F)) & \xrightarrow{c_i(3)} & H^{6-i}(\Gamma(F,3)) \\
 & \searrow \qquad \nearrow {\scriptstyle c_i(N)} & \\
 & H_i(GL_N(F)) &
\end{array}
$$

and Im $c_i(N) =$ Im $c_i(3)$.

Recall that $H_n(GL_n(F)) = H_n(GL(F))$ (see [S1]), so

$$K_n(F)_{\mathbb{Q}} = \text{Prim } H_n(GL(F),\mathbb{Q}) = \text{Prim } H_n(GL_n(F),\mathbb{Q}) \ .$$

Put

$$K_n^{(j)}(F)_{\mathbb{Q}} := \text{Im}\,(H_n(GL_{n-j}(F),\mathbb{Q}) \to H_n(GL_n(F),\mathbb{Q})) \cap \text{Prim } H_n(GL_n(F),\mathbb{Q}) \ .$$

$$K_n^{[j]}(F)_{\mathbb{Q}} := K_n^{(j)}(F)_{\mathbb{Q}} / K_n^{(j+1)}(F)_{\mathbb{Q}} \ .$$

Conjecture 3.2 (A.A. Suslin, unpublished) $K_n^{[j]}(F)_{\mathbb{Q}} \cong gr_{\gamma}^{n-j} K_n(F)_{\mathbb{Q}}$.

So we get canonical homomorphisms

$$C_i^{[i-3]} : K_i^{[i-3]}(F)_{\mathbb{Q}} \longrightarrow H^{6-i}(\Gamma(F,3) \otimes \mathbb{Q}) \quad (i = 3,4,5) \ .$$

A.A. Suslin proved that $K_n^{[0]}(F)_{\mathbb{Q}} \cong K_n^M(F)_{\mathbb{Q}}$. So $C_3^{[0]}$ is an isomorphism. $C_4^{[1]}$ and $C_5^{[2]}$ also should be isomorphisms. In any case $C_5^{[2]} : K_5^{[2]}(F) \to H^1(\Gamma(F,3) \otimes \mathbb{Q})$. We will construct a homomorphism $c_5 : K_5(F) \to H^1(\Gamma(F,3) \otimes \mathbb{Q})$ and show that the composition

$$K_5(\mathbb{C}) \xrightarrow{c_5} H^1(\Gamma(\mathbb{C},3) \otimes \mathbb{Q}) \xrightarrow{\tilde{\mathcal{L}}_3} R$$

coincides with Borel regulator [Bo2]. This implies immediately Zagier's conjecture about $\zeta_F(3)$.

3. Construction of homomorphism 3.2. Choose a volume form $\omega \in \wedge^3(V^3)^*$. Set $\Delta(l_i, l_j, l_k) := \langle \omega, l_i \wedge l_j \wedge l_k \rangle \in F^*$. Put

$$f_4(3) : (l_1, \dots, l_4) \mapsto \text{Alt } \Delta(l_1, l_2, l_3) \wedge \Delta(l_1, l_2, l_4) \wedge \Delta(l_1, l_3, l_4) \ . \qquad (3)$$

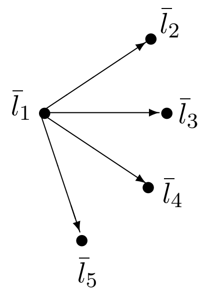
Here

$$\text{Alt } f(l_1, \dots, l_n) := \sum_{\sigma \in S_n} (-1)^{|\sigma|} f(l_{\sigma(1)}, \dots, f_{\sigma(n)}) \ .$$

Lemma 3.3 $f_4(3)$ *does not depend on the choice of* ω.

Proof. Let $\omega' = \lambda\omega$, $\lambda \in F^*$. Then the difference between the right-hand sides of (3.3) computed using ω' and ω is $\text{Alt}(\lambda \wedge \Delta(l_1, l_2, l_4) \wedge \Delta(l_1, l_3, l_4))$. But this is 0 because we alternate an expression that is symmetric with respect to permutation of 1 and 4 □.

For a vector $l \in V^3$ let us denote by $\bar{l}$ the corresponding point in $P(V^3) = P^2$. Let us denote by $(\bar{l}_1 | \bar{l}_2, \dots, \bar{l}_4)$ the configuration of 4 points on P^1 obtained by projection of points $\bar{l}_2, \dots, \bar{l}_5$ with the center at point $\bar{l}_1$, see fig. 3.1 (All lines passing through $\bar{l}_1$ form a projective line; any point $m \neq \bar{l}_1$ defines a point on this line).



(fig. 3.1)

Now let $(m_1, \dots, m_4) \in C_4(2)$. Let us define the cross-ratio as $r(\bar{m}_1, \dots, \bar{m}_4)$ as follows

$$r(\bar{m}_1, \dots, \bar{m}_4) := \frac{\Delta(m_1, m_3)\Delta(m_2, m_4)}{\Delta(m_1, m_4)\Delta(m_2, m_3)} \ . \qquad (4)$$

It is clear that the right-hand side of (3.4) does not depend on length of m_i. We have

$$r(\bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{m}_4) \ = \ r(\bar{m}_2, \bar{m}_1, \bar{m}_3, \bar{m}_4)^{-1} = r(\bar{m}_1, \bar{m}_2, \bar{m}_4, \bar{m}_3)^{-1} =$$

$$= \quad 1 - r(\bar{m}_1, \bar{m}_3, \bar{m}_2, \bar{m}_4)\ . \tag{5}$$

The last equality is proved using the identity

$$\Delta(m_1, m_4)\Delta(m_2, m_3) - \Delta(m_1, m_2)\Delta(m_3, m_4) = \Delta(m_1, m_3)\Delta(m_2, m_4)\ .$$

Set

$$f_5(3)(l_1, \ldots, l_5) := \frac{1}{2}\mathrm{Alt}(\{r(\bar{l}_1|\bar{l}_2, \ldots, \bar{l}_5)\}_2 \otimes \Delta(l_1, l_2, l_3))\ . \tag{6}$$

Here $\{x\}_2$ means the image of $\{x\}$ in $\mathcal{B}_2(F)$.

Proposition 3.4 $f_5(3)$ *does not depend on* ω.

Proof. The difference between the right-hand sides of (3.6) computed using $\lambda \cdot \omega$ and ω is proportional to

$$\sum_{i=1}^{5}(-1)^i\{r(\bar{l}_i|\bar{l}_1, \ldots, \hat{\bar{l}}_i, \ldots, \bar{l}_5)\}_2 \otimes \lambda$$

because $\{r(m_1, \ldots, m_4)\}_2 \in \mathcal{B}_2(F)_{\mathbb{Q}}$ is sque-symmetric with respect to permutation of points m_i – see (3.5) and example 1 in s.4 of §1. So we need to prove the following

Lemma 3.5 *Let* $x_1, \ldots, x_5$ *be 5 points on* P^2 *in generic position. Then*

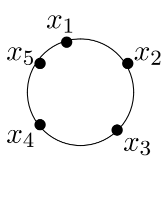
$$\sum_{i=1}^{5}(-1)^i\{r(x_i|x_1, \ldots, \hat{x}_i, \ldots, x_5)\} \in \mathcal{R}_2(F)\ .$$

This lemma follows from

Lemma 3.6 *Let* $m_1, \ldots, m_5$ *be 5 different points on* P^1. *Then*

$$R_2(m_1, \ldots, m_5) := \sum_{i=1}^{5}(-1)^i\{r(m_1, \ldots, \hat{m}_i, \ldots, m_5)\} \in \mathcal{R}_2(F)\ . \tag{7}$$

Indeed, let us consider a conic (a curve of order 2) passing through points $x_1, \ldots, x_5$ as a projective line. It remains to apply lemma 3.6 to these points on this projective line (see fig. 3.2)



(fig. 3.2)

Proof of lemma 3.5. Consider the following homomorphism of complexes

$$\begin{array}{ccccc} C_5(2) & \xrightarrow{d} & C_4(2) & \xrightarrow{d} & C_3(2) \\ & & \downarrow f_4(2) & & \downarrow f_3(2) \\ & & \mathbb{Z}[P^1_F] & \xrightarrow{\delta_2} & \wedge^2 F^* \end{array} \qquad (3.8)$$

$$f_3(2) : (l_1, l_2, l_3) \mapsto \Delta(l_1, l_2) \wedge \Delta(l_1, l_3) - \Delta(l_2, l_1) \wedge \Delta(l_2, l_3) +$$
$$+\Delta(l_3, l_1) \wedge \Delta(l_3, l_2)$$
$$f_4(2) : (l_1, \ldots, l_4) \mapsto \{r(\bar{l}_1, \ldots, \bar{l}_4)\} \ .$$

Direct calculation using (3.4) – (3.5) shows that (3.8) is commutative. So

$$\delta_2(\sum_{i=1}^{5}(-1)^i\{r(\bar{m}_1, \ldots, \hat{\bar{m}}_i, \ldots \bar{m}_5)\} \equiv \delta_2 \circ f_4(2) \circ d =$$
$$= f_3(2) \circ d^2 = 0 \ .$$

Now it is easy to complete the proof of lemma 3.6 using specialization □.

Proposition 3.7 $f_4(3) \circ d = d \circ f_5(3)$

Proof. Direct calculation using (3.4)□

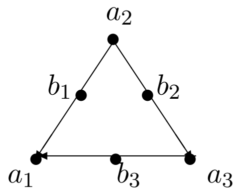
The main formula

$$f_6(3) : (l_1, \ldots, l_6) \mapsto \text{Alt}\left\{\frac{\Delta(l_1, l_2, l_4)\Delta(l_2, l_3, l_5)\Delta(l_3, l_1, l_6)}{\Delta(l_1, l_2, l_5)\Delta(l_2, l_3, l_6)\Delta(l_3, l_1, l_4)}\right\} \qquad (9)$$

4. The geometrical definition of the generalized cross-ratio (3.9). Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a configuration of 6 distinct points in P^2 such that a_1, a_2, a_3 does not lie on a line and $b_i \in \overline{a_i a_{i+1}}$ (see fig. 3.3). Let $P^2 = P(V_3)$. Choose vectors in V_3 such that they are projected to points a_i, b_i. By an abuse of notations we will denote them by the same letters. Choose $f_i \in V_3^*$ such that $f_i(a_i) = f_i(a_{i+1}) = 0$. Put

$$r_3'(a_1, a_2, a_3, b_1, b_2, b_3) = \frac{f_1(b_2) \cdot f_2(b_3) \cdot f_3(b_1)}{f_1(b_3) \cdot f_2(b_1) \cdot f_3(b_2)} \ . \qquad (10)$$

The right-hand side of (3.10) does not depend on the choice of vectors f_i, b_j.



(fig. 3.3)

Lemma 3.8 $r(b_1|a_2, a_3, b_2, b_3) = r'_3(a_1, a_2, a_3, b_1, b_2, b_3)$.

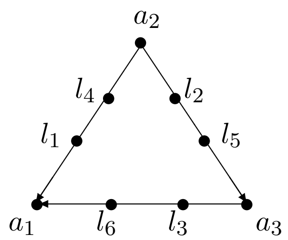
Proof. Put

$$f_1(v) := \Delta(b_1, a_2, v); f_2(v) := \Delta(b_2, a_3, v); f_3(v) := \Delta(b_3, a_3, v) .$$

Then the right-hand side of (3.10) is equal to

$$\frac{\Delta(b_1, a_2, b_2) \cdot \Delta(b_2, a_3, b_3) \cdot \Delta(b_3, a_3, b_1)}{\Delta(b_1, a_2, b_3) \cdot \Delta(b_2, a_3, b_1) \cdot \Delta(b_3, a_3, b_2)} = \frac{\Delta(b_1, a_2, b_2) \cdot \Delta(b_1, a_3, b_3)}{\Delta(b_1, a_2, b_3) \cdot \Delta(b_1, a_3, b_2)} = \\ = r(b_1|a_2, a_3, b_2, b_3) \qquad \Box$$

Now let $(l_1, \ldots, l_6)$ be a configuration of 6 distinct points in P^2 in generic position. Put $a_i := \overline{l_i l_{i+3}} \cap \overline{l_{i-1} l_{i+2}}$, $(1 \leq i \leq 3$, indices modulo 6; see fig. 3.4).



(fig. 3.4)

Then $l_i \in \overline{a_i a}_{i+1}$, so $(a_1, a_2, a_3, l_1, l_2, l_3)$ is a configuration of considered above type. Let us define the generalized cross-ratio $r_3 : C_6(3) \to \mathbb{Z}[P^1_F \backslash \{0, \infty\}]$ as follows:

$$r_3(l_1, \ldots, l_6) := \text{Alt}\ \{r'_3(a_1, a_2, a_3, l_1, l_2, l_3)\} \in \mathbb{Z}[P^1_F \backslash \{0, \infty\}] . \qquad (11)$$

More precisely, the alternation here means the following. Let $s \in S_6$ be a permutation and

$$a_i^{(s)} := \overline{l_{s(i)} l_{s(i+3)}} \cap \overline{l_{s(i-1)} l_{s(i+2)}} , \quad (1 \leq i \leq 3) .$$

Then

$$r_3(l_1,\ldots,l_6) := \sum_{s\in S_6}(-1)^{|\sigma(s)|}\{r_3'(a_1^{(s)},a_2^{(s)},a_3^{(s)},l_{s(1)},l_{s(2)},l_{s(3)})\}\ . \qquad (12)$$

Lemma 3.9 $r_3(l_1,\ldots,l_6) = f_6(3)(l_1,\ldots,l_6)$

Proof. It is sufficient to prove that

$$r_3'(a_1,a_2,a_3,l_1,l_2,l_3) = \frac{\Delta(l_1,l_2,l_4)\Delta(l_2,l_3,l_5)\Delta(l_3,l_1,l_6)}{\Delta(l_1,l_2,l_5)\Delta(l_2,l_3,l_6)\Delta(l_3,l_1,l_4)}\ .$$

But this follows immediately from the definition (3.10) if we put $f_i(v) := \Delta(l_i,l_{i+3},v)\ ,\quad i=1,2,3.$ $\square$

In the previous version of the proof of Zagier's conjecture about $\zeta_F(3)$ I used the same formulas for homomorphism $f_4(3)$ and $f_5(3)$, but a little bit different one for $f_6(3)$ that was not skew-symmetric. D. Zagier showed that formula 3.9 can be obtained by the skew-symmetrization of that formula.

5. Theorem 3.10 $f_5(3)\circ d = \delta\circ f_6(3)$.

Proof. Computing $\delta\circ f_6(3)$ using formula (3.9) and lemma (3.8) we get

$$\delta\circ f_6(3)(l_1,\ldots,l_6) = \mathrm{Alt}(\{r(l_1|l_2,l_3,l_4,a_3)\}_2\otimes\Delta(l_1,l_2,l_4)) =$$
$$= \frac{1}{2}\mathrm{Alt}([\{r(l_1|l_2,l_3,l_4,a_3)\}_2 - \{r(l_1|l_2,l_6,l_4,a_3)\}_2]\otimes\Delta(l_1,l_2,l_4))\ .$$

Here $a_3 = \overline{l_2l_5}\cap\overline{l_3l_6}$ and we understand alternation in the same way as in formula (3.11).

The 5-term relation for the configuration $(l_1|l_2,l_3,l_6,l_4,a_3)$ gives us

$$\delta\circ f_6(3)(l_1,\ldots,l_6) = \frac{1}{2}\mathrm{Alt}[-\{r(l_1|l_3,l_6,l_4,a_3)\}_2 + \{r(l_1|l_2,l_3,l_6,a_3)\}_2$$
$$-\{r(l_1|l_2,l_3,l_6,l_4)\}_2]\otimes\Delta(l_1,l_2,l_4))\quad (13)$$

Considering the projection onto the line $\overline{l_3l_6}$ we see that (see fig. 3.4)

$$(l_1|l_3,l_6,l_4,a_3)\equiv(l_4|l_3,l_6,l_1,a_3)$$
$$(l_1|l_2,l_3,l_6,a_3)\equiv(l_2|l_1,l_3,l_6,a_3)\ .$$

So the first 2 terms in the first factor in (3.13) disappear after alternation and we get

$$\delta\circ f_6(3)(l_1,\ldots,l_6) = -\frac{1}{2}\mathrm{Alt}(\{r(l_1|l_2,l_3,l_6,l_4)\}_2\otimes\Delta(l_1,l_2,l_4)) =$$
$$= -\frac{1}{2}\mathrm{Alt}(\{r(l_1|l_2,l_3,l_4,l_5)\}_2\otimes\Delta(l_1,l_2,l_3))\ . \quad (14)$$

But this coincides with $f_5(3) \circ d(l_1, \ldots, l_6)$ computed using formula (3.5) $\square$

6. The "7-term" functional equation for the trilogarithm.

Theorem 3.11

$$\sum_{i=1}^{7} (-1)^{i-1} \mathcal{L}_3(r_3(l_1, \ldots, \hat{l}_i, \ldots, l_7)) = 0 \ . \tag{15}$$

Proof. According to theorem 1.10 one has

$$\delta \circ f_6(3) \circ d = f_5(3) \circ d \circ d = 0 \ , \quad \text{i.e. (because } r_3 = f_6(3))$$
$$\delta \circ \left(\sum_{i=1}^{7} (-1)^{i-1} r_3(l_1, \ldots, \hat{l}_i, \ldots, l_7) \right) = 0 \quad \text{in } \mathcal{B}_2(F) \otimes F^* \ .$$

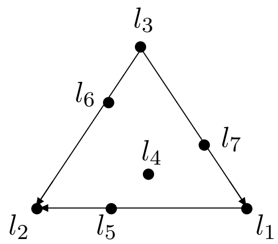
Apply theorem 2.1 in the case $n = 3$ we get

$$\sum_{i=1}^{7} (-1)^{i-1} \mathcal{L}_3(r_3(l_1, \ldots, \hat{l}_i, \ldots, l_7)) = \text{const.}$$

Using the specialization it is not hard to prove that this constant is zero (see, for example, explicit formula (3.17) below).

Remark. Our "7-term" functional equation has 840 summands. In order to get a shorter version we need to use a degenerate configurations $(l_1, \ldots, l_7)$. For example, let homogeneous coordinates of points l_i are represented by columns of the following matrix (see also fig. 3.5)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & c \\ 0 & 1 & 0 & 1 & a & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & b & 1 \end{bmatrix} \tag{16}$$



(fig. 3.5)

Put

$$R_3(a,b,c) := \oplus_{\text{cycle}} \left(\{ca - a + 1\} + \left\{\frac{ca-a+1}{ca}\right\} + \{c\} + \left\{\frac{(bc-c+1)}{(ca-a+1)b}\right\} - \right.$$
$$\left. \left\{\frac{ca-a+1}{c}\right\} + \left\{\frac{(bc-c+1)a}{(ca-a+1)}\right\} - \left\{\frac{(bc-c+1)}{(ca-a+1)bc}\right\} - \{1\} \right) \tag{17}$$
$$+\{-abc\} \ .$$

Here $\oplus_{\text{cycle}} f(a,b,c) := f(a,b,c) + f(c,a,b) + f(b,c,a)$. The functional equation (3.15) for this special configuration (3.16) has form

$$\mathcal{L}_3(R_3(a,b,c)) = 0 \ .$$

7. The Grassmanian bicomplex. This is the following bicomplex

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & C_{n+5}(n_2) & \xrightarrow{d} & C_{n+4}(n+2) & \xrightarrow{d} & C_{n+3}(n+2) & \\ & \downarrow d' & & \downarrow d' & & \downarrow d' & \\ \longrightarrow & C_{n+4}(n+1) & \xrightarrow{d} & C_{n+3}(n+1) & \xrightarrow{d} & C_{n+2}(n+1) & \\ & \downarrow d' & & \downarrow d' & & \downarrow d' & \\ \longrightarrow & C_{n+3}(n) & \xrightarrow{d} & C_{n+2}(n) & \xrightarrow{d} & C_{n+1}(n) & \end{array} \tag{18}$$

where

$$d' : (l_1, \ldots, l_m) \mapsto \sum_{i=1}^{m} (-1)^{i-1} (l_i | l_1, \ldots, \hat{l_i}, \ldots, l_m) \ .$$

Denote by $(T_*(n), \partial)$ the total complex associated with this bicomplex; $T_{n+1}(n) := C_{n+1}(n)$. Let us define a homomorphism $\psi_*(3)$

$$\begin{array}{ccccccc} \longrightarrow & T_6(3) & \longrightarrow & T_5(3) & \longrightarrow & T_4(3) & \\ & \downarrow \psi_6(3) & & \downarrow \psi_5(3) & & \downarrow \psi_4(3) & \\ & \mathcal{B}_3(F) & \longrightarrow & \mathcal{B}_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* & \end{array} \tag{3.19}$$

as follows. It coincides with homomorphism (3.2) on the subcomplex $C_*(n) \hookrightarrow T_*(n)$ and is zero on all other groups $C_*(n+i)$.

Theorem 3.12. *This is a correct definition, i.e.*

$$\psi_{3+i}(3) \circ d' = 0 \text{ for } i = 1,2,3 \ .$$

Proof.

a) $i = 1$. It is easy to see that

$$\psi_4(3) \circ d' : (l_1, \ldots, l_5) \mapsto \mathrm{Alt}\Delta(l_1, l_2, l_3, l_4) \wedge \Delta(l_1, l_2, l_3, l_5) \wedge \Delta(l_1, l_2, l_4, l_5).$$

The right-hand side is zero because we alternate an expression that is symmetric with respect to permutation of l_1 and l_2.

b) $i = 2$. The

$$\psi_5(3) \circ d' : (l_1, \ldots, l_6) \mapsto \frac{1}{2}\mathrm{Alt}\left(\{r(l_1, l_2 | l_3, l_4, l_5, l_6)\} \otimes \Delta(l_1, l_2, l_3, l_4)\right) .$$

This is zero for the same reason as above.

c) $i = 3$. We have to prove the following

$$\psi_6(3)(\sum_{i=1}^{7}(-1)^i \left(l_i | l_1, \ldots, \hat{l}_i, \ldots, l_7\right) = 0 . \tag{20}$$

This will be done in sections 8–9.

8. The duality of configurations (see §7 of [G2]). Let us denote by $\mathrm{Conf}_p(q)$ the set of all configurations of p vectors in a q-dimensional vector space V_q in generic position. There is a duality

$$* : \mathrm{Conf}_{m+n}(m) \to \mathrm{Conf}_{m+n}(n); \qquad *^2 = \mathrm{id}$$

that satisfies the following important properties:

1. $*$ commutes with the action of the permutation group S_{m+n} on vectors of a configuration.

2. If $*(l_1, \ldots, l_{m+n}) = (l'_1, \ldots, l'_{m+n})$, then

$$*(l_1, \ldots, \hat{l}_i, \ldots, l_{m+n}) = (l'_i | l'_1, \ldots, \widehat{l'_i}, \ldots, l'_{m+n})$$

i.e. the forgetting of the i-th vector of a configuration is dual to the projection along the i-th vector.

3. Let us choose volume forms in V^m and V^n; consider a partition

$$\{1, \ldots, m+n\} = \{i_1 < \ldots < i_m\} \cup \{j_1 < \ldots < j_n\} .$$

Then $\dfrac{\Delta(l_{i_1}, \ldots, l_{i_m})}{\Delta(l'_{j_1}, \ldots, l'_{j_n})}$ does not depend on a partition.

Three definitions of $*$: the Grassmanian, the coordinate, and the geometrical one, were suggested in §7 of [G2]. We need only the first two.

i) **The Grassmannian definition.** Let $(l_1, \ldots, l_{m+n})$ be a coordinate frame in a vector space V. Let us denote by $\hat{G}_m(V, \{e_i\})$ the set of all m-dimensional subspaces V that are in generic position to coordinate hyperplanes. R. MacPherson constructed in [Mac] an isomorphism $p : \hat{G}_m(V, \{e_i\}) \xrightarrow{\sim} \mathrm{Conf}_{m+n}(n)$. Namely, $p(h)$ is a configuration formed by images of l_i in V/h. Let $(f^1, \ldots, f^{m+n})$ be the dual basis in V^* and $h^\perp : \{f \in V^* | \langle f, v \rangle = 0 \text{ for any } v \in h\}$. Then the definition of $*$ is given by the following diagram

$$\begin{array}{ccc} \hat{G}_m(V, \{l_i\}) & \xrightarrow[\sim]{\perp} & \hat{G}_n(V^*, \{f^j\}) \\ p \wr \downarrow & & \downarrow \wr p \\ \mathrm{Conf}_{m+n}(n) & \xrightarrow{*} & \mathrm{Conf}_{m+n}(m) \end{array}$$

ii) **The coordinate definition.** A configuration of $(m+n)$ vectors in an m-dimensional coordinate space can be represented as columns of the following $m \times (m+n)$-matrix:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{11} & \ldots & a_{1n} \\ 0 & 1 & \ldots & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & a_{m1} & \ldots & a_{mn} \end{pmatrix} = (I_m, A) \ .$$

Then the dual configuration is represented by the $n \times (m+n)$-matrix $(-A^t, I_n)$. These definitions give the same duality. Indeed, the subspace h is generated by $l_{m+i} - \sum_{j=1}^{m} a_{ij} e_j$ and the subspace $h^\perp$ by $f^j + \sum_{i=1}^{n} a_{ij} f_{m+i}$.

Now properties 1), 2) follow immediately from the first definition, and 3) is easy to see from the second one.

9. The end of the proof of theorem 3.12c).

Proposition 3.13 $\psi_6(3)\left((l_1, \ldots, l_6) + *(l_1, \ldots, l_6)\right) = 0$.

Proof. If $*(l_1,\dots,l_6)=(l'_1,\dots,l'_6)$ then according to the property of $*$ we have

$$\frac{\Delta(l_1,l_2,l_4)\Delta(l_2,l_3,l_5)\Delta(l_3,l_1,l_6)}{\Delta(l_1,l_2,l_5)\Delta(l_2,l_3,l_6)\Delta(l_3,l_1,l_4)}=\frac{\Delta(l'_5,l'_6,l'_3)\Delta(l'_4,l'_6,l'_1)\Delta(l'_4,l'_5,l'_2)}{\Delta(l'_4,l'_6,l'_3)\Delta(l'_4,l'_5,l'_1)\Delta(l'_5,l'_6,l'_2)}\equiv$$
$$\equiv\frac{\Delta(l'_4,l'_5,l'_2)\Delta(l'_5,l'_6,l'_3)\Delta(l'_6,l'_4,l'_1)}{\Delta(l'_4,l'_5,l'_1)\Delta(l'_5,l'_6,l'_2)\Delta(l'_6,l'_4,l'_1)}\ .$$

But $\{x\}=\{x^{-1}\}\mathrm{mod}\mathcal{R}_3(F)_{\mathbb{Q}}$ and $(1,2,3,4,5,6)\mapsto(4,5,6,1,2,3)$ is an odd permutation, so proposition 3.13 is proved. □

Formula (3.19) and hence theorem 3.12 c) follows immediately from proposition 3.13 and property 2) of $*$ □

10. The bicomplex $C^m_*(n)$. Let us define a differential $d^{(k)}:\tilde{C}_p(n)\to\tilde{C}_{p-1}(n)$ as follows: $d^{(k)}:(\ell_1,\dots,\ell_p)\mapsto\sum_{i=1}^{p-k}(-1)^{i-1}(\ell_1,\dots,\hat{\ell}_{k+i},\dots,\ell_p)$. Note that $d^{(0)}\equiv d$ – see s.1.

Lemma 3.14 *The following complex is acyclic* ($k>0$):

$$\dots\longrightarrow\tilde{C}_{k+2}(n)\xrightarrow{d^{(k)}}\tilde{C}_{k+1}(n)\xrightarrow{d^{(k)}}C_k(n)\ .$$

The proof is in complete analogy with the one of Lemma 3.1.

Let $\mathrm{Sym}_k:\tilde{C}_p(n)\to\tilde{C}_p(n)$ be the symmetrisation of the first k vectors:

$$\mathrm{Sym}_k:(\ell_1,\dots,\ell_p)\mapsto\sum_{\sigma\in S_k}\frac{1}{k!}(x_{\sigma(1)},\dots,x_{\sigma(k)},x_{k+1},\dots,x_p)\ .$$

Define a homomorphism $\lambda^{(k)}:\tilde{C}_p(n)\to\tilde{C}_p(n)$ as follows:

$$\lambda^{(k)}:(\ell_1,\dots,\ell_p)\mapsto\sum_{i=1}^{p-k}(-1)^{i-1}\mathrm{Sym}_{k+1}(l_1,\dots,\hat{l}_{k+i},\dots,l_p)\ .$$

Lemma 3.15 $d^{(k+1)}\circ\lambda^{(k)}=-\lambda^{(k)}\circ d^{(k)}$.

Proof. It is obvious for the homomorphism $\tilde{\lambda}^{(k)}$ that is defined by the same formula as $\lambda^{(k)}$, but without symmetrisation. It remains to symmetrise the first $k+1$ vectors. □

Lemma 3.16 $\lambda^{(k+1)}\circ\lambda^{(k)}=0$.

Proof. Straightforward. (Note that $\tilde{\lambda}^{(k+1)} \circ \tilde{\lambda}^{(k)} \neq 0$.) $\square$

Therefore we get the following bicomplex $\tilde{C}^m * (n)$

$$\begin{array}{cccccccc} \ldots \longrightarrow & \tilde{C}_4(n) & \stackrel{d}{\longrightarrow} & \tilde{C}_3(n) & \stackrel{d}{\longrightarrow} & \tilde{C}_2(n) & \stackrel{d}{\longrightarrow} & \tilde{C}_1(n) \\ & \downarrow \lambda^{(1)} & & \downarrow \lambda^{(1)} & & \downarrow \lambda^{(1)} & & \downarrow \lambda^{(1)} \\ \ldots \longrightarrow & \tilde{C}_4(n) & \stackrel{d^{(1)}}{\longrightarrow} & \tilde{C}_3(n) & \stackrel{d^{(1)}}{\longrightarrow} & \tilde{C}_2(n) & \stackrel{d^{(1)}}{\longrightarrow} & \tilde{C}_1(n) \\ & \downarrow \lambda^{(2)} & & \downarrow \lambda^{(2)} & & \downarrow \lambda^{(2)} & & \\ \ldots \longrightarrow & \tilde{C}_4(n) & \stackrel{d^{(2)}}{\longrightarrow} & \tilde{C}_3(n) & \stackrel{d^{(2)}}{\longrightarrow} & \tilde{C}_2(n) & & \\ & \downarrow \lambda^{(3)} & & \downarrow \lambda^{(3)} & & & & \\ \ldots \longrightarrow & \tilde{C}_4(n) & \stackrel{d^{(3)}}{\longrightarrow} & \tilde{C}_3(n) & & & & \\ & \vdots & & & & & & \\ & \downarrow & & & & & & \\ \ldots \longrightarrow & \tilde{C}_{m-1}(n) & & & & & & \end{array} \tag{21}$$

Remark. The bicomplex $C^2_*(3)$ was considered by A.A. Suslin in §3 of [S3].

Let $(\tilde{\mathcal{D}}^m_*(n), \partial)$ be a complex, associated with the bicomplex $\tilde{C}^m_*(n)$. It is placed at degrees $-1, 0, +1, \ldots$, (∂ has degree -1).

Lemma 3.17 $H^i(\tilde{\mathcal{D}}^m_*(n)) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i \neq 0 \end{cases}$.

The proof follows immediately from lemmas 3.14 and 3.15.

The group $GL_n(F)$ acts naturally on the complex $\tilde{\mathcal{D}}^m_*(n)$. Let us denote complex $\tilde{\mathcal{D}}^m_*(n)_{GL_n(F)}$ as $\mathcal{D}^m_*(n)$. Lemma 3.17 implies that there is a canonical homomorphism

$$H_*(GL_n(F), \mathbb{Z}) \to H_*(\mathcal{D}^m_*(n)) \ .$$

Now let us define a homomorphism of complexes

$$\begin{array}{ccccccccc} \longrightarrow & \mathcal{D}^{(n-2)}_6(n) & \longrightarrow & \mathcal{D}^{(n-2)}_5(n) & \longrightarrow & \mathcal{D}^{(n-2)}_4(n) & \longrightarrow & & \\ & \downarrow f_3 & & \downarrow f_3 & & \downarrow f_3 & & & \\ \longrightarrow & T_6(3) & \longrightarrow & T_5(3) & \longrightarrow & T_4(3) & \longrightarrow & 0 \end{array} \tag{3.22}$$

More precisely, we will define a homomorphism $\tilde{f}$ of the corresponding bicomplex $C_*^{(n-2)}(n)$ to the Grassmanian bicomplex (see 3.18)

$$
\begin{array}{cccccc}
 & & & \downarrow & & \downarrow \\
 & & \longrightarrow & C_7(5) & \xrightarrow{d} & C_6(5) \\
 & \downarrow & & \downarrow d' & & \downarrow d' \\
 & C_7(4) & \xrightarrow{d} & C_6(4) & \xrightarrow{d} & C_5(4) \\
 & \downarrow d' & & \downarrow d' & & \downarrow d' \\
\longrightarrow & C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3)
\end{array}
$$

Namely, if $(l_1, \dots, l_m) \in C_m(p)$ is placed at the level k in the bicomplex $C_*^{n-2}(n)$, i.e. we apply to $(l_1, \dots, l_m)$ the horizontal differential $d^{(k)}$ (see (3.21) then we set

$$\tilde{f} : (l_1, \dots, l_m) \mapsto (l_1, \dots, l_k | l_{k+1}, \dots, l_m) \in C_{m-k}(p-k)$$

Here we use the following notations. Let $(l_1, \dots, l_k, \dots, l_m) \in C_m(V)$. Let us denote by $\langle l_1, \dots, l_n \rangle$ the subspace generated by $l_1, \dots, l_k$. Then

$$(l_1, \dots, l_k | l_{k+1}, \dots, l_m)$$

is the configuration of $m-k$ vectors in $V / \langle l_1, \dots, l_k \rangle$.

So we get a homomorphism f_3 of the corresponding total complexes (see (3.22)). The composition of this homomorsphism with homomorphism ψ constructed above

$$
\begin{array}{ccccccc}
 & \longrightarrow & T_6(3) & \longrightarrow & T_5(3) & \longrightarrow & T_4(3) \\
 & & \downarrow \psi_6(3) & & \downarrow \psi_5(3) & & \downarrow \psi_4(3) \\
0 & \longrightarrow & \mathcal{B}_3(F) & \longrightarrow & \mathcal{B}_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^*
\end{array}
$$

gives the desired homomorphism of complexes

$$
\begin{array}{ccccccc}
 & \longrightarrow & \mathcal{D}_6^{(n-2)}(n) & \longrightarrow & \mathcal{D}_5^{(n-2)}(n) & \longrightarrow & T_4(3) \\
 & & \downarrow \psi \circ f & & \downarrow \psi \circ f & & \downarrow \psi \circ f \\
0 & \longrightarrow & \mathcal{B}_3(F) & \longrightarrow & \mathcal{B}_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^*
\end{array}
$$

Therefore we get the canonical homomorphisms

$$H_i(GL_n(F)) \longrightarrow H^{6-i}(\Gamma(F, 3)) \; . \tag{23}$$

Lemma 3.18 *The restriction of the homomorphisms (3.23) to the subgroup $H_i(GL_3(F))$ coincide with the one (3.3).*

Proof. Choose $n-3$ linearly independent vectors $v_1, \ldots, v_{n-3}$ in an n-dimensional vector space V_n and a 2-dimensional complementary subspace $V_3 : V_n = \langle v_1, \ldots, v_{n-3} \rangle \oplus V_3$. Then there is a homomorphism of complexes $\xi : C_*(V_3) \longrightarrow \mathcal{D}_*^{n-2}(V_n)$ where $\xi(C_*(V_3))$ lies in the lowest line of the bicomplex (3.20) and $\xi : (l_1, \ldots, l_k) \mapsto (v_1, \ldots, v_{n-3}, l_1, \ldots, l_k)$.

It is clear from the definition that we get a commutative diagram

$$\begin{array}{ccc} C_*(3) & \xrightarrow{\ \xi\ } & \mathcal{D}_*^{(n-2)}(n) \\ & {\scriptstyle (3.2)} \searrow \qquad \swarrow {\scriptstyle \varphi \circ f} & \\ & \Gamma(F, 3) & \end{array}$$

□

Finally, the restriction of the homomorphisms

$$c_i(3) : H_i(GL_3(F)) \longrightarrow H^{6-i}(\Gamma(F; 3))$$

to the image of the subgroup $H_i(GL_2(F))$ is equal to zero, because the resolution $\tilde{D}_*(3)$ of the trivial $GL_3(F)$-module $\mathbb{Z}$ has a $GL_2(F)$- invariant section

$$\begin{array}{ccccc} & & & & \mathbb{Z} \\ & & & & \downarrow \\ \ldots & \longrightarrow & \tilde{C}_2(3) & \longrightarrow & \tilde{C}_1(3) \end{array}$$

Namely, if $V_3 = V_2 \oplus \langle v \rangle$, then the formula $n \mapsto n \cdot (v) \in \tilde{C}_1(3)$ defines a $GL_2(V_2)$-invariant section $\mathbb{Z} \to \tilde{C}_*(V_3)$.

So we have constructed homomorphisms

$$\begin{array}{rcl} C_5^{[2]} : K_5^{[2]}(F)_{\mathbb{Q}} & \longrightarrow & H^1(\Gamma(F;3)_{\mathbb{Q}}) \\ C_4^{[1]} : K_4^{[1]}(F)_{\mathbb{Q}} & \longrightarrow & H^2(\Gamma(F;3)_{\mathbb{Q}}) \, . \end{array}$$

Conjecture 3.19 *Homomorphism $C_4^{[1]}, C_5^{[2]}$ are isomorphisms.*

11. Explicit formula for a 5-cocycle representing a class of continuous cohomology of $GL_3(\mathbb{C})$. Choose a point $x \in \mathbb{C}P^2$. Then there is a measurable cocycle

$$f^{(x)} : \underbrace{GL_3(\mathbb{C}) \times \ldots \times GL_3(\mathbb{C})}_{6 \text{ times}} \to R$$

$$f^{(x)}(g_1, \ldots, g_6) := \mathcal{L}_3(r_3(g_1x, \ldots, g_6x)) \qquad (24)$$

where r_3 is the generalized cross-ration of 6 points in P^2 (see s. 4). It is certainly invariant under the left action of $GL_3(\mathbb{C})$. So the 7-term relation (3.15) for the trilogarithm just means that $f^{(x)}$ is a measurable cocycle of $GL_3(\mathbb{C})$. Different points x gives cohomologous cocycles.

The function $\mathcal{L}_3(z)$ is continuous on $\mathbb{C}P^1$ and hence bounded. So the function $f^{(x)}$ is also bounded. Applying proposition 1.14 from ch. III of [Gu] we see that the cohomology class of the cocycle (3.24) lies in

$$Im(H^5_{\text{cts}}(GL_3(\mathbb{C}), R) \longrightarrow H^5(GL_3(\mathbb{C}), R)) \ .$$

It remains to be proved that the constructed class coincides with the Borel class in $H^5_{\text{cts}}(GL_3(\mathbb{C}), R)$. Several possible proofs were suggested in [G2]. In §5 a different proof will be given. It is based on an explicit formula for indecomposable element in $H^6_{\mathcal{D}}(BGL_3(\mathbb{C})_\bullet, R(3))$

4 Some arguments for the main conjecture

1. We have already seen before that $L(F)^\vee_{-1}$ must be isomorphic to $F^*_{\mathbb{Q}}$.

2. The Bloch-Suslin complex. Let us define a subgroup $R_2(F) \subset \mathbb{Z}[P^1_F \backslash 0, 1, \infty]$ as follows:

$$R_2(F) := \left\{ \sum_{i=0}^{4} (-1)^i \{r(x_0, \cdots, \hat{x}_i, \cdots, x_4)\} \ , \quad x_i \in P^1_F \ , \quad x_i \neq x_j \right\} \ .$$

Then $\delta_2(R_2(F)) = 0$ according to lemma 3.6 ($\delta_2 : \{x\} \mapsto (1-x) \wedge x$). So we get a complex $B_F(2)$ (the Bloch-Suslin complex)

$$B_2(F) \xrightarrow{\delta} \Lambda^2 F^* \ , \quad B_2(F) := \frac{\mathbb{Z}[P^1_F \backslash 0, 1, \infty]}{R_2(F)} \qquad (25)$$

where the group $B_2(F)$ placed in degree 1 and δ has degree +1. Let $K_3^{\text{ind}}(F) := \text{Coker}\ (K_3^M(F) \to K_3(F))$. Using some ideas of S. Bloch, A.A. Suslin proved the following remarkable theorem (see also closely related results of J. Dupont and S.-H. Sah [DS] [Sa]).

Theorem 4.1 [S2] *There is an exact sequence*

$$0 \longrightarrow \mathrm{Tor}(F^*, F^*)^{\sim} \longrightarrow K_3^{\mathrm{ind}}(F) \longrightarrow H^1(B_F(2)) \longrightarrow 0$$

where $\mathrm{Tor}(F^*, F^*)^{\sim}$ *is the unique nontrivial extension of* $\mathbb{Z}/2\mathbb{Z}$ *by* $\mathrm{Tor}(F^*, F^*)$.

In particular,

$$H^1(B_F(2)_{\mathbb{Q}}) \cong K_3^{\mathrm{ind}}(F)_{\mathbb{Q}} \cong K_3^{[1]}(F)_{\mathbb{Q}} \cong gr_j^2 K_3(F)_{\mathbb{Q}} \ .$$

So the complex $B_F(2)$ has the same homology as the complex $L(F)^{\vee}_{-2} \stackrel{\partial}{\longrightarrow} \Lambda^2 L(F)^{\vee}_{-1}$. Assume that there is a homomorphism of complexes

$$\begin{array}{ccc} B_2(F) & \stackrel{\delta}{\longrightarrow} & \Lambda^2 F^* \\ \varphi_2 \downarrow & & \| \\ L(F)^{\vee}_{-2} & \stackrel{\partial}{\longrightarrow} & \Lambda^2 F^* \end{array} \qquad (4.2)$$

that induces isomorphism on cohomologies modulo torsion. Then
$\varphi_2 : B_2(F) \longrightarrow L(F)^{\vee}_{-2}$ must be an isomorphism.

In fact, the existence of a homomorphism of complexes (4.2) can be deduced from results of [BGSV], [BMS] and standard assumptions about the category $\mathcal{M}_T(F)$. After this, using the Borel theorem, one can prove that the induced homomorphism $H^1(B_F(2)_{\mathbb{Q}}) \longrightarrow H^1_{(2)}(L(F)_{\bullet})$ must be an isomorphism for number fields. Finally, the rigidity conjecture tells us that the same is true for an arbitrary field F (see s.12 of §1 in [Go2]).

Note that theorem 4.1 and isomorphism $K_3^{\mathrm{ind}}(F) \cong K_3^{\mathrm{ind}}(F(t))$ imply that the canonical map $B_2(F) \longrightarrow \mathcal{B}_2(F)$, $(\{x\} \mapsto \{x\})$ is an isomorphism.

3. Weight 3 motivic complexes. Recall that the generalized cross-ratio $r_3 : C_6(P^2) \longrightarrow \mathbb{Z}[P^1_F]$ is defined by the following formula

$$r_3(l_1, \cdots, l_6) = \mathrm{Alt}\left\{ \frac{\Delta(l_1,l_2,l_4)\Delta(l_2,l_3,l_5)\Delta(l_3,l_1,l_6)}{\Delta(l_1,l_2,l_5)\Delta(l_2,l_3,l_6)\Delta(l_3,l_1,l_4)} \right\} \ .$$

Set

$$R_3(F) := \left\{ \sum_{i=0}^{6} (-1)^i r_3(l_0, \cdots, \hat{l}_i, \cdots, l_6) \ , \quad \text{where } (l_0, \cdots, l_6) \in C_7(P^2) \right\}$$

$$B_3(F) := \mathbb{Z}[P^1_F]/R_3(F), \{0\}, \{\infty\} \ .$$

Theorem 3 implies that $\delta_3(R_3(F)) = 0$, so we get a complex $B_F(3)$:

$$B_3(F) \xrightarrow{\delta} B_2(F) \otimes F^* \xrightarrow{\delta} \Lambda^3 F^*$$

where $B_3(F)$ placed in degree 1 and δ has degree $+1$.

Let us assume that there is a homomorphism $\varphi_3 : B_3(F) \longrightarrow L(F)^\vee_{-3}$ making the following diagram commutative (we have assumed $L(F)^\vee_{-2} \cong B_2(F)_{\mathbb{Q}}$, $\quad L(F)^\vee_{-1} \cong F^*_{\mathbb{Q}}$):

$$\begin{array}{ccc} B_3(F) & \xrightarrow{\delta} & B_2(F) \otimes F^* \\ \downarrow{\scriptstyle \varphi_3} & & \wr\downarrow{\scriptstyle \varphi_2 \otimes \varphi_1} \\ L(F)^\vee_{-3} & \xrightarrow{\partial} & L(F)^\vee_{-2} \otimes L(F)^\vee_{-1} \end{array}$$

Then we get a morphism of complexes

$$\begin{array}{ccccc} B_3(F) & \xrightarrow{\delta} & B_2(F) \otimes F^* & \xrightarrow{\delta} & \Lambda^3 F^* \\ \downarrow{\scriptstyle \varphi_3} & & \wr\downarrow{\scriptstyle \varphi_2 \otimes \varphi_1} & & \wr\downarrow{\scriptstyle \Lambda^3\varphi_1} \\ L(F)^\vee_{-3} & \xrightarrow{\partial} & L(F)^\vee_{-2} \otimes L(F)^\vee_{-1} & \xrightarrow{\partial} & \Lambda^3 L(F)^\vee_{-1} \end{array}$$

The bottom complex is just $(\Lambda^\bullet_{(3)}(L(F)_\bullet), \partial)$: the part of grading 3 of the cochain complex of the Lie algebra $L(F)_\bullet$.

The results of §3 give considerable evidence for the expected isomorphisms

$$H^i(B_F(3)_{\mathbb{Q}}) \cong H^i(\Lambda^\bullet_{(3)}(L(F)_\bullet) \qquad (3)$$

(According to conjecture 3.19 and ((1.3) both sides are isomorphic to $K^{[3-i]}_{6-i}(F)_{\mathbb{Q}}$). (4.3) implies that $\varphi_3 : B_3(F)_{\mathbb{Q}} \longrightarrow L(F)^\vee_{-3}$ is an isomorphism. I expect, of course, that $B_3(F)_{\mathbb{Q}} \cong \mathcal{B}_3(F)_{\mathbb{Q}}$.

In any case the complexes $(\Lambda^\bullet_{(n)}(L(F)_\bullet), \partial)$ for $n = 1, 2, 3$ look like the complexes $\Gamma(F; n)$. But already the weight 4 part of the cochain complex of $L(F)_\bullet$, that is

$$L(F)^\vee_{-4} \xrightarrow{\partial} \oplus \begin{matrix} L(F)^\vee_{-3} \otimes L(F)^\vee_{-1} \\ \Lambda^2 L(F)^\vee_{-2} \end{matrix} \xrightarrow{\partial} L(F)^\vee_{-2} \otimes \Lambda^2 L(F)^\vee_{-1} \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \Lambda^4 L(F)^\vee_{-1} \qquad (4)$$

looks quite different from $\Gamma(F;4)$, because we have an extra term $\Lambda^2 L(F)^\vee_{-2}$ $(4 = 2+2)$ that has no analog in $\Gamma(F;4)$. So assuming a homomorphism $\varphi_4 : \mathcal{B}_4(F)_\mathbb{Q} \longrightarrow L(F)^\vee_{-4}$ making (2.1b) commutative we get a morphism of complexes $\tilde{\varphi}_4 : \Gamma(F;4) \longrightarrow (\Lambda^\bullet_{(4)}(L(F)_\bullet), \partial)$, but it cannot be an isomorphism. However

Theorem 4.2 *$\tilde{\varphi}_4 : H^3\Gamma(F;4) \longrightarrow H^3(\Lambda^\bullet_{(4)}(L(F)_\bullet), \partial)$ is an isomorphism.*

Proof. Set

$$\begin{aligned}
&\kappa(x,y) := \varphi_3\left[-\{1-x\} - \{1-y\} + \left\{\frac{1-x}{1-y}\right\} - \left\{\frac{1-x^{-1}}{1-y^{-1}}\right\}\right] \otimes \frac{x}{y} \\
&\varphi_3\{x\} \otimes (1-y) - \varphi_3\{y\} \otimes (1-x) + \varphi_3\{\frac{x}{y}\} \otimes \frac{1-x}{1-y} \qquad (5) \\
&-\varphi_2\{x\} \wedge \varphi_2\{y\}
\end{aligned}$$

that lies in

$$L(F)^\vee_{-3} \otimes L(F)^\vee_{-1} \oplus \Lambda^2 L(F)^\vee_{-2} = B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)\ .$$

Lemma 4.3 $\partial(\kappa(x,y)) = 0$.

Proof. Direct calculation. $\square$

Note that

$$\kappa(x,y) + \varphi_2\{x\} \wedge \varphi_2\{y\} \subset (\varphi_3 \otimes \varphi_1)(B_3(F) \otimes F^*) = L(F)^\vee_{-3} \otimes L(F)^\vee_{-1}\ .$$

So it follows from lemma 4.3 that

$$\partial(\Lambda^2 L(F)^\vee_{-2}) \subset \partial(L(F)^\vee_{-3} \otimes L(F)^\vee_{-1})\ .$$

But this is the only fact that we need in order to prove theorem 4.2 $\square$

Corollary 4.4 *Assume that for $n = 1,2,3$ we have isomorphisms $\varphi_n : B_n(F)_\mathbb{Q} \xrightarrow{\sim} L(F)^\vee_{-n}$ making diagram (2.1b) commutative. Then*

$$H^{n-1}_{(n)}(L(F)_\bullet) \cong \frac{\mathrm{Ker}(B_2(F)_\mathbb{Q} \otimes \Lambda^{n-2}F^*_\mathbb{Q} \longrightarrow \Lambda^n F^*_\mathbb{Q})}{\{x\}_2 \otimes x \wedge \Lambda^{n-3}F^*_\mathbb{Q}}\ .$$

Proof. The left-hand side is just the cohomology of the following complex

$$\oplus \quad \begin{matrix} L^\vee_{-3} \otimes \Lambda^{n-3} L^\vee_{-1} \\ \Lambda^2 L^\vee_{-2} \otimes \Lambda^{n-4} L^\vee_{-1} \end{matrix} \xrightarrow{\partial} L^\vee_{-2} \otimes \Lambda^{n-2} L^\vee_{-1} \xrightarrow{\partial} \Lambda^n L^\vee_{-1}\ .$$

It remains to apply theorem 4.2 $\Box$

Lemma 4.3 tells us that an element $\varphi_4(x,y) \in L(F)^\vee_{-4}$ should exist such that

$$\partial\varphi_4(x,y) = \kappa(x,y)$$

(The reason is that $\Gamma(F,n)_{\mathbb{Q}}$ should be a "resolution" for $K^M_n(F)$. See appendix in [G2].) Let us assume that such $\varphi_4(x,y)$ exists.

5. Weight 5 motivic complexes. The part of grading 5 of the cochain complex of $L(F)_\bullet$ looks as follows:

$$L^\vee_{-5} \xrightarrow{\partial} \oplus \begin{matrix} L^\vee_{-4}\otimes L^\vee_{-1} \\ L^\vee_{-3}\otimes L^\vee_{-2} \end{matrix} \xrightarrow{\partial} \oplus \begin{matrix} L^\vee_{-3}\otimes \Lambda^2 L^\vee_{-1} \\ \Lambda^2 L^\vee_{-2}\otimes L^\vee_{-1} \end{matrix} \xrightarrow{\partial} L^\vee_{-2}\otimes\Lambda^3 L^\vee_{-1} \xrightarrow{\partial} \Lambda^5 L^\vee_{-1} \ .$$

We would like to prove that the component $\partial_{3,2} : L^\vee_{-5} \longrightarrow L^\vee_{-3}\otimes L^\vee_{-2}$ of the coboundary ∂ is an epimorphism. Unfortunately it is not quite clear how to construct an element in $L^\vee_{-5}$ because $L^\vee_{-5}$ itself is a quite mysterious object. However, assuming the existence of $\phi_4(x,y)$ we can find an element in $L^\vee_{-4}\otimes L^\vee_{-1}\oplus L^\vee_{-3}\otimes L^\vee_{-2}$ with zero coboundary, whose component in $L^\vee_{-3}\otimes L^\vee_{-2}$ is $\varphi_3\{x\}\otimes\varphi_2\{y\}$. We expect that such a cycle should be in $\varphi(L^\vee_{-5})$.

Let us do this. We assume a $\varphi_4 : \mathcal{B}_4(F) \longrightarrow L(F)^\vee_{-4}$ making (2.1b) commutative. Consider the following element

$$\phi_5(x,y) := \phi_4(x,y)\otimes\frac{x}{y} + \varphi_4\left\{\frac{x}{y}\right\}\otimes\frac{1-x}{1-y} + \varphi_4\{x\}\otimes(1-y) + \\ +\varphi_4(y)\otimes(1-x) - \varphi_3\{x\}\otimes\varphi_2\{y\} - \varphi_3\{y\}\otimes\varphi_2\{x\} \ . \qquad (6)$$

Lemma 4.5 $\partial\phi_5(x,y) = 0$.

Proof. Direct calculations using formula (4.5) for $\partial\phi_4(x,y) = \kappa(x,y)$.

The $L^\vee_{-3}\otimes L^\vee_{-2}$ component of $-1/2(\phi_5(x,y)+\phi_5(x,y^{-1}))$ is equal to $\varphi_3\{x\}\otimes\varphi_2\{y\}$ because $\{y\}_2+\{y^{-1}\}_2 = 0$ in $B_2(F)_{\mathbb{Q}}$ and $\{y\}_3 = \{y^{-1}\}_3$ in $B_3(F)_{\mathbb{Q}}$.

We can pursue this idea further and "construct" by induction elements $\phi_n(x,y) \in L(F)^\vee_{-n}$ (using the same assumptions as above) such that

$$\partial\phi_n(x,y) = \phi_{n-1}(x,y)\otimes\frac{x}{y} + \varphi_{n-1}\left\{\frac{x}{y}\right\}\otimes\frac{1-x}{1-y} + \qquad (4.7)_{(n)} \\ +\sum_{k=1}^{[\frac{n}{2}]}(-1)^{k-1}(\varphi_{n-k}\{x\}\otimes\varphi_k\{y\} + (-1)^{n-k}\varphi_{n-k}\{y\}\otimes\varphi_k\{x\})$$

for n odd; for n even we have the same formula, but the last term will be $(-1)^{n/2-1}\varphi_{n/2}\{x\} \wedge \varphi_{n/2}\{y\}$. (Here $\varphi_1(a) := 1 - a \in F^*$).

Proposition 4.6 *Suppose that $\partial\phi_{n-1}(x,y)$ is given by formula (4.7)$_{(n-i)}$. Then the coboundary of the right hand side of (4.7)$_{(n)}$ is equal to* 0.

Proof. Direct calculation using the formula

$$\partial(\phi_{n-1}(x,y) \otimes \frac{x}{y} + \varphi_{n-1}\left\{\frac{x}{y}\right\} \otimes \frac{1-x}{1-y}) =$$
$$\sum_{k=1}^{[\frac{n}{2}]}(-1)^{k-1}(\varphi_{n-k}\{x\} \otimes \varphi_k\{y\} + (-1)^{n-k}\varphi_{n-k}\{y\} \otimes \varphi_k\{x\})$$

(for n odd the last term in this sum should be $(-1)^{\frac{n-1}{2}-1}\varphi_{\frac{n-1}{2}}\{x\} \wedge \varphi_{\frac{n-1}{2}}\{y\}$.

6. Nonexistence of natural generators for $L(F)_{\leq -2}$ inside $L(F)_\bullet$. Let us choose a splitting $s : B_4^\vee \longrightarrow L_{-4}$ of the exact sequence

$$0 \longrightarrow [L_{-2}, L_{-2}] \longrightarrow L_{-4} \overset{s}{\underset{\longrightarrow}{\longleftarrow}} B_4^\vee \longrightarrow 0$$

This means that we make a choice of degree -4 generators for $L(F)_{\leq -2}$. Then the composition of the commutator map $L_{-3} \otimes L_{-1} \longrightarrow L_{-4}$ with the projection of L_{-4} along $s(B_4^\vee)$ gives us a homomorphism

$$L_{-3} \otimes L_{-1} \longrightarrow \wedge^2 L_{-2} \ .$$

Assume that $L(F)_{-i} = B_i(F)^\vee$ for $i = 1, 2, 3$. Then dualising we get a homomorphism

$$p : B_2(F) \wedge B_2(F) \longrightarrow B_3(F) \otimes F^* \ . \tag{7}$$

The following result, proved in collaboration with D. Zagier, shows that there are no any such reasonable non-zero map! More precisely, let us call a map p *natural* if it is given by the following formula

$$p : \{x\}_2 \wedge \{y\}_2 \mapsto \sum_i \{\varphi_i(x,y)\}_3 \otimes \psi_i(x,y) \tag{8}$$

where $\varphi_i(x,y)$ and $\psi_i(x,y)$ are rational functions with coefficients in $\mathbb{Q}$.

Theorem 4.7 *There are no natural non-zero homomorphism* (4.8).

Proof. In the case $F = \mathbb{C}$ there is a homomorphism

$$l : B_3(\mathbb{C}) \otimes \mathbb{C}^* \longrightarrow B_2(\mathbb{C}) \otimes \mathbb{C}^* \otimes \mathbb{C}^* \longrightarrow R$$
$$l : \{z_1\}_3 \otimes z_2 \mapsto \mathcal{L}_2(z_1) \cdot \log|z_1| \cdot \log|z_2| \ .$$

Consider the composition

$$B_2(\mathbb{C}) \wedge B_2(\mathbb{C}) \xrightarrow{p} B_3(\mathbb{C}) \otimes \mathbb{C}^* \xrightarrow{l} R$$
$$l \circ p : \{x\}_2 \wedge \{y\}_2 \mapsto \sum_i \mathcal{L}_2(\varphi_i(x,y)) \cdot \log|\varphi_i(x,y)| \cdot \log|\psi_i(x,y)| \ . \qquad (9)$$

The right-hand side of (4.10) satisfies the 5-term functional equation on variable x (as well as on y) because both p and l are homomorphisms and so $l \circ p(R_2(\mathbb{C}) \wedge \{y\}_2) = 0$. From the other hand we have the following beautiful result of S. Bloch [Bl1])

Theorem 4.8 *Let $f(z)$ be a measurable function satisfying the 5-term relation $\sum_{i=1}^5 (-1)^i \mathcal{L}_2(r(x_1, \ldots, \hat{x}_i, \ldots, x_5)) = 0$. Then $f(z) = \lambda \cdot \mathcal{L}_2(z)$ for some $\lambda \in \mathbb{C}$.* □

Applying this theorem to the right-hand side of (4.10) considered as a function on x and then as a function on y we get

$$\sum_i \mathcal{L}_2(\varphi_i(x,y)) \cdot \log|\varphi_i(x,y)| \cdot \log|\psi_i(x,y)| = \lambda \cdot \mathcal{L}_2(x) \cdot \mathcal{L}_2(y) \ . \qquad (10)$$

The left expression is skewsymmetric on x, y because of its definition (4.10), while $\lambda \cdot \mathcal{L}_2(x) \cdot \mathcal{L}_2(y)$ is obviously symmetric. So $\lambda = 0$.

There is another argument: the right-hand side of (4.11) is invariant under the involution $x \mapsto \bar{x}, y \mapsto \bar{y}$, while the left one is skew invariant. (It works for a homomorphism $\tilde{p} : B_2 \otimes B_2 \longrightarrow B_3 \otimes F^*$). Therefore $\lambda = 0$.

This is the crucial point and now it becomes absolutely clear that theorem 4.7 is true. However we will present a rigorous proof.

Let us choose a generic number $y_0 \in \mathbb{C}$. There is a natural basis $(x - a)$, $a \in \mathbb{C}$ in the $\mathbb{Q}$-vector space $\mathbb{C}(x)^* /_{\mathbb{C}^*} \otimes \mathbb{Q}$. Using this basis we can rewrite (4.9) as follows ($\alpha \in \mathbb{C}^*$):

$$\begin{aligned} \sum_i \{\varphi_i(x,y_0)\}_3 \otimes \psi_i(x,y_0) &= \sum_{i,j} n_j^i \{f_i^j(x)\}_3 \otimes (x - a_i) + \\ &+ \sum_j n_j^0 \{f_0^j(x)\}_3 \otimes \alpha \ . \end{aligned}$$

Then (4.11) looks like $\sum_i A_i(x) + A_0(x)$ where

$$A_i(x) := \sum_{i,j} n^i_j \mathcal{L}_2(f^j_i(x)) \cdot \log |f^j_i(x)| \cdot \log |x - a_i| \,. \tag{11}$$

The function $\mathcal{L}_2(z)$ is real-analytical on $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$, continuous on $\mathbb{C}P^1$ and has a singularity of type $r \cdot \log r$ at $z = 0, 1, \infty$. Therefore for any $k > 0$ the functions $A_k(x)$ and $A_{\neq k}(x) := \sum_{i \neq k} A_i(x) + A_0(x)$ have the following singularity near $x = a_k$:

$$\begin{aligned} A_k(x) &: r^{2m} \log^{m+1} r \quad \text{or} \quad r^m \log^{m+2} r \qquad\qquad (m \geq 0) \\ A_{\neq k}(x) &: r^{2m} \log^{m} r \quad \text{or} \quad r^m \log^{m+1} r \qquad\qquad (m \geq 1) \end{aligned}$$

For example, if $f^j_k(x) = 1 - c \cdot (x - a_k)^m + \cdots$ then $A_k(x)$ has singularity of type $r^{2m} \log^{m+1} r$. Fortunately all pairs $(2m, m+1)$, $m \geq 0$; $(m, m+2), m \geq 0$; $(2m, m), m \geq 1; (m, m+1), m \geq 1$ are different. (For example $(2m, m+1) = (m, m+1)$ only if $m = 0$, but in our situation $m \geq 1$ for $(m, m+1)$.) This means that the singularities of $A_k(x)$ never coincide with the one of $A_{\neq k}(x)$ and hence $A_k(x) + A_{\neq k}(x) = 0$ implies

$$\sum_j n^k_j \mathcal{L}_2(f^j_k(x)) \cdot \log |f^j_k(x)| \equiv 0 \,. \tag{12}$$

Now let us prove that

$$\sum_j n^k_j \{f^j_k(x)\}_2 \otimes f^j_k(x) = 0 \quad \text{in} \quad B_2(\mathbb{C}(x)) \otimes \mathbb{C}(x)^* \,.$$

Let us decompose this element using our basis in $\mathbb{C}(x)^*/_{\mathbb{C}^*}$:

$$\begin{aligned} &\sum n^k_j \{f^j_k(x)\}_2 \otimes f^j_k(x) = \\ &= \sum_{m,n} c^m_n \{g^n_m(x)\}_2 \otimes (x - b_m) + \sum c^0_n \{g^n_0(x)\}_2 \otimes \beta \,. \end{aligned}$$

Then (4.13) looks like

$$\sum_{m,n} c^m_n \mathcal{L}_2(f^j_k(x)) \cdot \log |x - b_m| + \sum c^0_n \mathcal{L}_2(g^n_0(x)) \cdot \log |\beta| = 0 \,.$$

Looking on the type of singularities of this expression near $x = b_m$ it is easy to see that for any m

$$\sum_n c^m_n \mathcal{L}_2(g^n_m(x)) \equiv 0 \,.$$

Proposition 4.9 *If* $\sum_n c_n \mathcal{L}_2(f_n(x)) \equiv 0$ *for some* $f_n(x) \in \mathbb{C}(x)$ *then*

$$\sum_n c_n \{f_n(x)\}_2 - \sum_n c_n \{f_n(0)\}_2 = 0 \text{ in } B_2(\mathbb{C}) \ .$$

Proof. Let

$$\delta_2(\sum_n c_n \{f_n(x)\}_2) = \sum_i (x-\alpha_i) \wedge (x-\beta_i) + \sum_j \delta_j \wedge (x-\gamma_j) + \sum_i \varepsilon_i \otimes \xi_i \ .$$

Then

$$\begin{aligned} 0 &= d(\sum_n c_n \mathcal{L}_2(f_n(x))) = \sum_i -(\log|x-\alpha_i| \cdot d\arg(x-\beta_i) + \\ &+ \log|x-\beta_i| d\arg(x-\alpha_i)) - \sum_j \log|\delta_j| d\arg(x-\gamma_i) \ . \end{aligned}$$

Now look on singularity of the right-hand side at $x = \alpha_i$. The first term has singularity of type $\log r$, while $d\arg(x-\alpha_i)$ has different type of singularity because

$$d\arg z = \frac{-ydx + xdy}{x^2+y^2} \ , \quad z = x+iy \ .$$

Therefore $\delta_2(\sum_n c_n \{f_n(x)\}_2) = 0$ and so by definition

$$\sum_n c_n \{f_n(x)\}_2 - \sum_n c_n \{f_n(0)\}_2 \in R_2(\mathbb{C}) \ . \tag{13}$$

□

Let us decompose the element $(\delta_3 \otimes id) \circ p(\{x\}_2 \wedge \{y_0\}_2)$ using the basis $(x-b_j) \otimes (x-a_i)$, $(x-b_j) \otimes \alpha_i$, $\beta_j \otimes (x-\alpha_i)$, $\beta_j \otimes \alpha_i$ in $\mathbb{C}(x)^*_{\mathbb{Q}} \otimes \mathbb{C}(x)^*_{\mathbb{Q}}$:

$$(\delta_3 \otimes id) \circ p(\{x\}_2 \wedge \{y_0\}_2) = \sum_{i,j} (\alpha_{ij})_2 \otimes (x-b_j) \otimes (x-a_i) + \cdots$$

where $(\alpha_{ij})_2 \in B_2(\mathbb{C})$. Insert into this formula the 5-term relation

$$\{x\}_2 - \{z\}_2 + \{z/x\}_2 - \left\{\frac{1-x^{-1}}{1-z^{-1}}\right\}_2 + \left\{\frac{1-x}{1-z}\right\}_2$$

instead of $\{x\}_2$. It is easy to see that for generic $z \in \mathbb{C}$ $(x-b_j) \otimes (x-a_i)$ will appear with coefficient $(\alpha_{ij})_2$. Hence $(\alpha_{ij})_2 = 0$. Pursuing further this argument we get

$$(\delta_3 \otimes id) \circ p(\{x\}_2 \wedge \{y_0\}) = 0 \quad \text{in} \quad B_2(\mathbb{C}(x)) \otimes \mathbb{C}(x)^* \otimes \mathbb{C}(x)^* \ .$$

So for any $x_0 \in \mathbb{C}$

$$p(\{x\}_2 \wedge \{y_0\}_2) - p(\{x_0\}_2) = 0 \quad \text{in} \quad B_3(\mathbb{C}) \otimes \mathbb{C}^* \ .$$

The same argument with the 5-term relation as above shows that in fact $p(\{x\}_2 \wedge \{y_0\}_2) = 0$. Using this it is easy to complete the proof of theorem 4.7. $\square$

7. Recall that one of the Beilinson-Lichtenbaum axioms predicts existence of the tensor product of motivic complexes $\Gamma(n) \overset{L}{\otimes} \Gamma(m) \longrightarrow \Gamma(n+m)$ defined in the derived category. Theorem 4.7 implies that for our complexes $\Gamma(F;n)_{\mathbb{Q}}$ natural tensor product exists as a morphism in the derived category *only* and cannot be defined at the level of complexes even for $m = n = 2$.

Indeed, an essential ingredient of construction of a natural morphism of complexes

$$\begin{array}{c} [(B_2 \xrightarrow{\delta} \wedge^2 F^*) \otimes (B_2 \xrightarrow{\delta} \wedge^2 F^*)] \\ \Big\downarrow m_{2,2} \\ [B_4 \xrightarrow{\delta} B_3 \otimes F^* \xrightarrow{\delta} B_2 \otimes \wedge^2 F^* \xrightarrow{\delta} \wedge^4 F^*] \end{array}$$

is the existence of the following commutative diagram

$$\begin{array}{ccc} B_2 \otimes B_2 & \xrightarrow{\delta \otimes id - id \otimes \delta} & B_2 \otimes \wedge^2 F^* \oplus \wedge^2 F^* \otimes B_2 \\ \Big\downarrow m_{2,2}^{(2)} & & \Big\downarrow m_{2,2}^{(3)} \\ B_3 \otimes F^* & \xrightarrow{\delta} & B_2 \otimes \wedge^2 F^* \end{array} \qquad (4.15)$$

But $m_{2,2}^{(2)}$ must be zero by theorem 4.7 and $m_{2,2}^{(3)}$ should equal to $(id, id \circ s)$ where s is the switch, so (4.15) cannot be commutative.

I am completely sure there is the same situation with tensor products of complexes $\Gamma(F, *)$ for any $m \geq 2, n \geq 2$.

Notice that we have a natural homomorphism

$$\delta(k) : B_n \longrightarrow B_{n-k} \otimes \underbrace{F^* \otimes \ldots \otimes F^*}_{k \text{ times}}$$
$$\delta(k) := (\delta \otimes id) \circ \delta(k-1) \ ; \quad \delta(1) := \delta \ .$$

Conjecture 4.10 *The only nontrivial natural homomorphisms $\otimes_i B_i \longrightarrow \otimes_j B_j$ are (up to a permutation) tensor products of the homomorphisms $\delta(k)$.*

Finally look at the tensor product $\Gamma(1) \otimes \Gamma(1) \longrightarrow \Gamma(2)$, i.e. $F^* \otimes F^* \longrightarrow \Gamma(2)$. Theorem 4.1 suggests that it should be defined in the derived category: $F^* \overset{L}{\otimes} F^* \longrightarrow \Gamma(2)$, providing $\mathrm{Tor}(F^*, F^*) \subset H^1(\Gamma(2))$.

5 Explicit formulas for the universal Chern class $c_3 \in H^6_?(BGL_{3\bullet}, \mathbb{Q}(3))$ in motivic and Deligne cohomology

1. The third motivic complex $\Gamma(X;3)$ for a regualr scheme (see s. 14 of §1 in [G2]). Let F be a field with a discrete valuation v and the residue class $\bar{F}_v (= \bar{F})$. The group of units U has a natural homomorphism $U \longrightarrow \bar{F}^*$, $u \mapsto \bar{u}$. An element $\pi \in F^*$ is prime if $\mathrm{ord}_v \pi = 1$. Let us construct a canonical homomorphism of complexes

$$\partial_v : \Gamma(F, n) \longrightarrow \Gamma(\bar{F}_v, n-1)[-1] \qquad (25)$$

such that the induced homomorphism

$$H^n(\Gamma(F, n)) = K_n^M(F) \longrightarrow H^{n-1}(\Gamma(\bar{F}_v, n-1)) = K_{n-1}^M(\bar{F}_v)$$

coincides with Milnor's tame symbol on $K_n^M(F)$.

There is a homomorphism $\theta : \wedge^n F^* \longrightarrow \wedge^{n-1} \bar{F}_v^*$ uniquely defined by the following properties ($u_i \in U$):

1. $\theta\ (\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \bar{u}_1 \wedge \cdots \wedge \bar{u}_{n-1}$.

2. $\theta\ (u_1 \wedge \cdots \wedge u_n) = 0$.

It clearly does not depend on the choice of π.

Let us define a homomorphism $s_v : \mathbb{Z}[P^1_F] \longrightarrow \mathbb{Z}[P^1_{\bar{F}_v}]$ as follows

$$s_v\{x\} = \begin{cases} \{\bar{x}\} & \text{if } x \text{ is a unit} \\ 0 & \text{otherwise} \end{cases} .$$

Then it induces a homomorphism (see s. 9 §1 of [G2])

$$s_v : \mathcal{B}_k(F) \longrightarrow \mathcal{B}_k(\bar{F}_v)) .$$

Set

$$\partial_v := s_v \otimes \theta : \mathcal{B}_k(F) \otimes \wedge^{n-k} F^* \longrightarrow \mathcal{B}_k(\bar{F}_v) \otimes \wedge^{n-k-1} \bar{F}_v^* .$$

Lemma 5.1 *The homomorphism ∂_v commutes with the coboundary δ and hence defines a homomorphism of complexes* (5.1).

See s. 14 of §1 in [G2] $\square$

Now let X be an arbitrary regular scheme, $X^{(i)}$ the set of all codimension i points of X, $F(x)$ the field of functions corresponding to a point $x \in X^{(i)}$. We define the third motivic complex $\Gamma(X;3)$ as the total complex associated with the following bicomplex:

$$\begin{array}{ccccccc}
\wedge^3 F(X)^* & \xrightarrow{\partial_1} & \coprod_{x\in X^{(1)}} \wedge^2 F(x)^* & \xrightarrow{\partial_2} & \coprod_{x\in X^{(2)}} F(x)^* & \xrightarrow{\partial_3} & \coprod_{x\in X^{(3)}} \mathbb{Z} \\
\uparrow \delta & & \uparrow \delta & & & & \\
\mathcal{B}_2(F(X)) \otimes F(X)^* & \xrightarrow{\partial_1} & \coprod_{x\in X^{(1)}} \mathcal{B}_2(F(X)) & & & & \\
\uparrow \delta & & & & & & \\
\mathcal{B}_3(F(X)) & & & & \partial = \oplus\partial_{v_x} & &
\end{array} \tag{5.2}$$

where $\mathcal{B}_3(F(X))$ placed in degree 1 and coboundaries have degree $+1$.

The coboundaries ∂_i are defined as follows. $\partial_1 := \coprod_{x\in X^{(1)}} \partial_{v_x}$. The others are a little bit more complicated. Let $x \in X^{(k)}$ and $v_1(y), \dots, v_m(y)$ be all discrete valuations of the field $F(x)$ over a point $y \in X^{(k+1)}, y \in \bar{x}$. Then $\overline{F(x)_i} := \overline{F(x)}_{v_i(y)} \supset F(y)$. (If $\bar{x}$ is nonsingular at the point y, then $\overline{F(x)_i} = F(y)$ and $m = 1$). Let us define a homomorphism $\partial_2 : \wedge^2 F(x) \longrightarrow F(y)^*$ as the composition

$$\wedge^2 F(x)^* \xrightarrow{\oplus\partial_{v_i(y)}} \oplus_{i=1}^m \overline{F(x)_i}^* \xrightarrow{\oplus N_{\overline{F(x)_i}/F(y)}} F(y)^*$$

and $\partial_3 : F(x)^* \longrightarrow \coprod_{y\in X^{(3)}} \mathbb{Z}$ as the composition

$$F(x)^* \xrightarrow{\oplus\partial_{v_i}} \oplus_{i=1}^m \mathbb{Z} \xrightarrow{\sum} \mathbb{Z}\ .$$

2. Explicit formula for the motivic Chern class $c_3 \in H^6_{\mathcal{M}}(BGL_3(F)_\bullet, \mathbb{Z}(3))$. Set $G^n := \underbrace{G \times \ldots \times G}_{n \text{ times}}$. Recall that

$$BG_\bullet := pt \underset{s_1}{\overset{s_0}{\leftleftarrows}} G \underset{s_2}{\overset{s_0}{\Lleftarrow}} G^2 \underset{s_3}{\overset{s_0}{\Lleftarrow}} G^3 \underset{s_4}{\overset{s_0}{\Lleftarrow}} \cdots$$

is the simplicial scheme representing the classifying space of the group G. There is canonical G-bundle over $BG_\bullet$ (G acts on the left on $EG_\bullet$).

$$\begin{array}{ccccccccc} EG_\bullet & & G & \leftleftarrows & G^2 & \Lleftarrow & G^3 & \cdots & G^4 \; \cdots \\ G \Big\downarrow \pi & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ BG_\bullet & & pt & \leftleftarrows & G^1 & \Lleftarrow & G^2 & \cdots & G^3 \; \cdots \end{array} \tag{5.3}$$

The cochain we have to construct lives in the following bicomplex (we will show a part on diagram 5.4 and the remaining one on (5.5))

$$\begin{array}{ccc} \vdots & & \\ \cdots & & \\ \uparrow \delta & & \\ \wedge^3 F(G^3)^* \oplus (\coprod) & \xrightarrow{s^*} & \wedge^3 F(G^4) \oplus \coprod_{x \in (G^4)^{(1)}} \mathcal{B}_2(F(x)) \\ & & \uparrow \delta \\ & & \mathcal{B}_2(F(G^4)) \otimes F(G^4)^* \xrightarrow{s^*} \mathcal{B}_2(F(G^5)) \otimes F(G^5)^* \\ & & \qquad\qquad\qquad \uparrow \delta \\ & & \qquad\qquad\qquad \mathcal{B}_3(F(G^5)) \xrightarrow{s^*} 0 \end{array} \tag{5.4}$$

Here $s^* := \sum(-1)^i s_i^*$, and $(\coprod) := \coprod_{x \in (G^3)^{(1)}} \mathcal{B}_2(F(x))$.

Let $v \in V^3$, where V^3 is a three dimensional vector space over F. Put (see section 3 of §3)

$$\begin{aligned} m_0(g_1, \ldots, g_5) &:= r_3(v, g_1 v, \ldots, g_5 v) \in \mathcal{B}_3(F(G^5)) \\ m_1(g_1, \ldots, g_4) &:= -f_5(3)(v, g_1 v, \ldots, g_4 v) \in \mathcal{B}_2(F(G^4)) \otimes F(G^4)^* \\ m_2(g_1, g_2, g_3) &:= f_4(3)(v, g_1 v, g_2 v, g_3 v) \in \wedge^3 F(G^3)^* \end{aligned}$$

Theorem 5.2 a) $s^* m_0 = 0$
b) $s^* m_1 + \delta m_0 = 0$
c) $s^* m_2 + \delta m_1 = 0$.

Proof. a) follows from the definition of $B_3(F)$ and existence of the homomorphism $B_3(F) \longrightarrow \mathcal{B}_3(F)$.
b) is equivalent to theorem 3.10.
c) follows from proposition 3.7 and the following simple but important remark: $\Delta(l_1, l_2, l_3)$ appears in formula (3.7) with factor $\{r(l_4|l_1, l_2, l_3, l_4)\}_2$ that is zero if $\Delta(l_1, l_2, l_3) = 0$. (This implies that $\mathcal{B}_2(F(x))$-component of δm_1 is zero for any $x \in (G^4)^1$) □

We see that this part of construction of cocycle c_3 is essentially equivalent to a construction of a homomorphism of complexes (3.2). The remaing part of bicomplex (5.4) looks as follows:

$$\begin{array}{ccccc}
\coprod_{x\in G^{(3)}} \mathbb{Z} & & & & \\
\uparrow \partial_3 & & & & \\
\coprod_{x\in G^{(2)}} F(x)^* & \stackrel{s^*}{\longrightarrow} & \coprod_{x\in (G^2)^{(2)}} F(x)^* & & \\
 & & \uparrow & & \\
 & & \coprod_{x\in (G^2)^{(1)}} F(x)^* & \stackrel{s^*}{\longrightarrow} & \coprod_{x\in (G^3)^{(1)}} \wedge^2 F(x)^* \\
 & & & & \uparrow \partial_1 \\
 & & & & \wedge^3 F(G^3)^* \oplus (\coprod) \quad \vdots
\end{array} \qquad (5.5)$$

Let us describe the corresponding components of the cocycle c_3. Put

$$\mathcal{D}_{v,1} = \{(g_1, g_2) \in G \times G | \Delta(v, g_1 v, g_2 v) = 0\} .$$

For generic $(g_1, g_2) \in \mathcal{D}_{v,1}$ we have $\dim\langle v, g_1 v, g_2 v\rangle = 2$, so we can set

$$\begin{aligned} m_3(g_1, g_2) \quad := \quad & -6(\Delta_2(v, g_1 v) \wedge \Delta_2(v, g_2 v) - \Delta_2(g_1 v, v) \wedge \Delta_2(g_1 v, g_2 v) \\ + \quad & \Delta_2(g_2 v, v) \wedge \Delta_2(g_2 v, g_1 v)) \in \wedge^2 F(\mathcal{D}_{v,1})^* . \end{aligned}$$

(Δ_2 is defined using a volume form in $\langle v, g_1 v, g_2 v\rangle$).

Lemma 5.3. $s^* m_3 + \partial_1 m_2 = 0$.

Proof. This is equivalent to the following: $\Delta(l_0, l_1, l_2)$ appears in formula

$$f_4(3)(l_0, l_1, l_2, l_3) = \mathrm{Alt}\Delta(l_0, l_1, l_2) \wedge \Delta(l_0, l_1, l_3) \wedge \Delta(l_0, l_2, l_3)$$

with factor

$$3f_3(2)(l_3|l_0,l_1,l_2) := 6(\Delta(l_0,l_1,l_3)\wedge\Delta(l_0,l_2,l_3) -$$
$$-\Delta(l_1,l_0,l_3)\wedge\Delta(l_1,l_2,l_3)+\Delta(l_2,l_0,l_3)\wedge\Delta(l_2,l_1,l_3))\Box$$

Set $\mathcal{D}_{v,2} = \{g \in G | gv = \lambda v$ for some $\lambda \in F^*\}$. We have canonical invertable function $\lambda(g) := \frac{gv}{v}$ on $\mathcal{D}_{v,2}$. Put $m_4(g) := 6 \cdot \lambda(g)$.

Lemma 5.4. $s^* m_4 + \partial_2 m_3 = 0$; $\partial_3 m_4 = 0$.

Proof. In complete analogy with the previous one. $\Box$

So we have constructed the cocycle $(m_0(g_1,\ldots,g_5),\ldots,m_4(g))$ representing a class $c_3 \in H^6_{\mathcal{M}}(BGL_3(F)_\bullet, \mathbb{Z}(3))$. In the next section for any complex algebraic manifold X a regulator

$$R_3 : H^\bullet_{\mathcal{M}}(X,\mathbb{Z}(3)) \longrightarrow H^\bullet_{\mathcal{D}}(X,R(3))$$

will be constructed. We will apply it to c_3.

3. Explicit construction of the regulator R_3. Recall that a (real-valued) p-current on X is by definition a linear continuous functional on the space of $(\dim_R X - p)$-forms with compact support. Let us denote by $\mathcal{A}^p_X$ the space of all p-currents on X. There is a differential $d : \mathcal{A}^p_X \longrightarrow \mathcal{A}^{p+1}_X$, and the
de Rham complex $(A^\bullet_X, d)$ is a resolution of the constant sheaf R.

The third Deligne complex $\underline{\tilde{R}}(3)_X$ can be defined as a total complex associated with the following bicomplex (see [B3]):

$$\begin{array}{ccccccccccc}
\mathcal{A}^0_X & \xrightarrow{d} & \mathcal{A}^1_X & \xrightarrow{d} & \mathcal{A}^2_X & \xrightarrow{d} & \mathcal{A}^3_X & \xrightarrow{d} & \mathcal{A}^4_X & \xrightarrow{d} & \ldots \\
 & & & & & & \uparrow{\scriptstyle Re} & & \uparrow{\scriptstyle -Re} & & \\
 & & & & & & \Omega^3_X & \xrightarrow{\partial} & \Omega^4_X & \xrightarrow{\partial} & \ldots
\end{array}$$

Here $\mathcal{A}^0_X$ placed in degree 1 and $(\Omega^\bullet_X, \partial)$ is the de Rham complex of holomorphic forms.

The Deligne complex $\underline{\tilde{R}}(n)_X$ is defined as follows:

$$\underline{\tilde{R}}(n)_X := \text{Cone}\,(\Omega^{\geq n}_X \xrightarrow{\alpha_n} \mathcal{A}^\bullet_X)[-1]$$

where $\alpha_n = (-1)^{n-1}\cdot Re$ for odd n and $(-1)^n Im$ for even.

To compute $H^*(X, \underline{\tilde{R}}(n)_X)$ we will use the Dolbeaux resolution $(\mathcal{A}^{\geq p,q}_X)$ for the complex of sheaves $(\Omega^{\geq n}_X, \partial)$ where $\mathcal{A}^{p,q}_X$ is the space of copmlex-valued (p,q)-currents.

Example 5.5 $\frac{dz}{z} \in \mathcal{A}^{1,0}_{\mathbb{C}}$ and $\overline{\partial}\left(\frac{dz}{z}\right) = 2\pi i\delta(0)dz\overline{dz}$. So $\overline{\partial}\log f = 2\pi i\delta(f)df\overline{df}$ for $f \in \mathcal{O}_X$.

Example 5.6 $d\arg f \in \mathcal{A}^1_X$ and $d(d\arg f) = 2\pi i\delta(f)df d\overline{f}$ for $f \in \mathcal{O}_X$.

In order to produce the regulator R_3 we will construct maps (that are not homomorphisms of complexes, see however proposition 5.7 below)

$$s_n : \Gamma(Spec\mathbb{C}(X); n) \longrightarrow \tilde{R}(n)_X \quad (n \leq 3)\ .$$

Namely, the map $s_3(\cdot)$

$$\begin{array}{ccccccc}
\mathcal{B}_3(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathcal{B}_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \wedge^3\mathbb{C}(X)^* & \longrightarrow & 0 \\
\downarrow{\scriptstyle s_3(1)} & & \downarrow{\scriptstyle s_3(2)} & & \downarrow{\scriptstyle s_3(3)\oplus d\log\wedge d\log\wedge d\log} & & \\
\mathcal{A}^0_X & \xrightarrow{d} & \mathcal{A}^1_X & \xrightarrow{(d,0)} & \mathcal{A}^2_X \oplus \Omega^3_X & \longrightarrow & \dots
\end{array} \tag{5.6}$$

is defined as follows:

$$\begin{array}{l}
s_3(1) : \{f(x)\}_3 \mapsto \mathcal{L}_3(f(x)) \\
s_3(2) : \{f(x)\}_2 \otimes g(x) \mapsto -\mathcal{L}_2(f(x))d\arg g(x) + \\
\qquad +\frac{1}{3}\log|g(x)| \cdot (\log|1-f(x)|d\log|f(x)| - \log|f(x)|d\log|1-f(x)|) \\
s_3(3) : f_1 \wedge f_2 \wedge f_3 \mapsto \text{Alt}\left(\frac{1}{2} \cdot \log|f_1|d\arg f_2 \wedge d\arg f_3 - \right. \\
\qquad \left. -\frac{1}{6} \quad \cdot |f_1|d\log|f_2|d\log|f_3|\right) \in \mathcal{A}^2_X; \\
d\log \wedge d\log \wedge d\log : f_1 \wedge f_2 \wedge f_3 \mapsto d\log f_1 \wedge d\log f_2 \wedge d\log f_3 \in \Omega^3_X
\end{array}$$

Proposition 5.7 *Then maps $s_3(\cdot)$ define a homomorphism of complexes*

$$\begin{array}{ccccc}
\mathcal{B}_3(\mathbb{C}(X)) & \longrightarrow & \mathcal{B}_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \longrightarrow & \wedge^3\mathbb{C}(X)^* \\
\downarrow{\scriptstyle {}_3(1)} & & \downarrow{\scriptstyle s_3(2)} & & \downarrow{\scriptstyle s_3(3)} \\
S^0_{\eta(X)} & \xrightarrow{d} & S^1_{\eta(X)} & \xrightarrow{d} & S^2_{\eta(X)}
\end{array} \tag{5.7}$$

where $S^p_{\eta(X)}$ is the space of p-forms at the generic point $\eta(X)$ of X.

Proof. Direct calculation using (1.14). □

This proposition means that $s_3(\cdot)$ is a homomorphism of complexes modulo currents supported on subvarieties of nonzero codimension of X.

The map

$$\begin{array}{ccc} \mathcal{B}_2(\mathbb{C}(X)) & \xrightarrow{\delta} & \wedge^2\mathbb{C}(X)^* \\ \downarrow{\scriptstyle s_2(1)} & & \downarrow{\scriptstyle s_2(2)\oplus d\log\wedge d\log} \\ \mathcal{A}^0_X & \xrightarrow{(d,0)} & \mathcal{A}^1_X \oplus \Omega^2_X \end{array} \qquad (5.8)$$

is defined as follows:

$$\begin{aligned} s_2(1) &\ :\ \{f(x)\}_2 \mapsto \mathcal{L}_2(f(x)) \\ s_2(2) &\ :\ f\wedge g \mapsto -\log|f| d\arg g + \log|g| d\arg f \in \mathcal{A}^1_X\ . \end{aligned}$$

Finally, $s_1 : f(x) \mapsto [\log|f(x)|, -\frac{df}{f}] \in \mathcal{A}^0_X \oplus \Omega^1_X$.

If $i : Y \hookrightarrow X$ is a complex algebraic subvariety of codimension d then there is a canonical homomorphism of complexes $i_* : \underline{\tilde{R}}(m)_Y \longrightarrow \underline{\tilde{R}}(m+d)_X$ provided by natural maps $i_* : \mathcal{A}^{p,q}_Y \hookrightarrow \mathcal{A}^{p+d,q+d}_X$. Therefore there is a collection of maps

$$i_* \circ s_{n-d} : \coprod_{x\in X^{(d)}} \Gamma(\mathrm{Spec}\mathbb{C}(X), n-d) \longrightarrow \underline{\tilde{R}}(n)_X\ . \qquad (9)$$

Recall that by definition $\Gamma(X,3)$ is the total complex associated with the following bicomplex

$$\Gamma(\mathrm{Spec}\ \mathbb{C}(X),3) \xrightarrow{\partial_1} \coprod_{x\in X^{(1)}} \Gamma(\mathrm{Spec}\ \mathbb{C}(x),2)[-1] \xrightarrow{\partial_2} \coprod_{x\in X^{(2)}} \mathbb{C}(x)^*[-2]$$
$$\xrightarrow{\partial_3} \coprod_{x\in X^{(3)}} \mathbb{Z}[-3]\ . \qquad (10)$$

So applying (5.9) to this complex we get the desired map

$$R_3 : \Gamma(X,3) \longrightarrow \tilde{R}(3)_X\ . \qquad (11)$$

Theorem 5.8 (5.11) *is a homomorphism of complexes.*

Proof. Follows immediately from the construction and proposition 5.7 together with analogous claim for s_2 and examples (5.5), (5.6). $\square$

Remark 5.9 We can define regulators $R_n : \Gamma(X,n) \longrightarrow \tilde{R}(n)_X$ in complete analogy with this definition of R_3. The only thing that we need is an explicit formula for $s_n(\cdot)$. See [G3] for details and formulas.

4. Formula for a cocycle representing $c_3 \in H^6_{\mathcal{D}}(BGL_3(\mathbb{C})_\bullet, R(3))$. Let v be a vector in a 3-dimensional vector space V^3, $G = GL(V^3)$. Cocycle $c_3^{(v)}$ we have to construct will depend on v and look as follows:

$$\begin{array}{cccccccccc}
(f^4_{(v)}, w^{3,2}_{(v)}) & \xrightarrow{s^*} & \uparrow{\scriptstyle\overline{\partial}} & & & & & & & \\
 & & (f^3_{(v)}, w^{3,1}_{(v)}) & \xrightarrow{s^*} & \uparrow{\scriptstyle\overline{\partial}} & & & & & \\
 & & & & (f^2_{(v)}, w^{3,0}_{(v)}) & \xrightarrow{s^*} & \uparrow{\scriptstyle d} & & & \\
 & & & & & & f^1_{(v)} & \xrightarrow{s^*} & \uparrow{\scriptstyle d} & \\
 & & & & & & & & f^0_{(v)} & \xrightarrow{s^*} 0 \\
\hline
pt & \Leftarrow & G^1 & \Lleftarrow & G^2 & \cdots & G^3 & \cdots & G^4 \cdots G^5 & \cdots
\end{array} \tag{5.12}$$

Set (see (5.4), (5.6)):

$$\begin{aligned}
f^1_{(v)}(g_1, \ldots, g_5) : &= \mathcal{L}_3(m^0_{(v)}(g_1, \ldots, g_5)) := \mathcal{L}_3(r_3(v, g_1 v, \ldots g_5 v)) \\
f^1_{(v)}(g_1, \ldots, g_4) &:= s_3(2)(m^1_{(v)}(g_1, \ldots, g_4)) \\
f^2_{(v)}(g_1, g_2, g_3) &:= s_3(3)(m^2_{(v)}(g_1, g_2, g_3)) \\
w^{(3,0)}_{(v)}(g_1, g_2, g_3) &:= d\log \wedge d\log \wedge d\log(m^2_{(v)}(g_1, g_2, g_3)) \\
f^3_{(v)}(g_1, g_2) &:= i_{1*} s_2(2)(m^3_{(v)}(g_1, g_2)) \\
w^{3,1}_{(v)}(g_1, g_2) &:= i_{1*} d\log \wedge d\log(m^3_{(v)}(g_1, g_2)) \\
f^4_{(v)}(g) &:= i_{2*} d\log(m^4_{(v)}(g))
\end{aligned}$$

Here $i_1 : \mathcal{D}_{v,1} \hookrightarrow G \times G$, $i_2 : \mathcal{D}_{v,2} \hookrightarrow G$ and

$$\begin{array}{rcl} i_{1*} & : & \Omega^2_{\mathcal{D}_{v,1}} \hookrightarrow \mathcal{A}^{3,1}_{G\times G}\ , \quad \mathcal{A}^1_{\mathcal{D}_{v,1}} \hookrightarrow \mathcal{A}^3_{G\times G} \\ i_{2*} & : & \Omega^1_{\mathcal{D}_{v,2}} \hookrightarrow \mathcal{A}^{3,2}_{G}\ , \quad \mathcal{A}^0_{\mathcal{D}_{v,2}} \hookrightarrow \mathcal{A}^4_{G}\ . \end{array}$$

Theorem 5.10 a) $c_3^{(v)}$ *is a cocycle.*
b) *It represents a nontrivial nondecomposable class in* $H^6_{\mathcal{D}}(BGL_3(\mathbb{C})_\bullet, R(3))$.

Proof. a) follows from theorem 5.2, lemmas 5.3 – 5.4 and theorem 5.8.

b) Let $\pi : EG_\bullet \longrightarrow BG_\bullet$ is the universal G-bundle realized as in (5.3). Then $EG_{(p)} = BG_{(p+1)}$ and so any i-cochain $c_{(\bullet)}$ for $BG_\bullet$ defines an $(i-1)$-cochain $\tilde{c}_{(\bullet)}$ for $EG_\bullet : \tilde{c}_{(p)} := c_{(p+1)}$. Moreover, if $c_{(0)} = 0$ and $c_{(\bullet)}$ is a cocycle then $d\tilde{c}_{(\bullet)} = c_{(\bullet)}$. Therefore $c_{(1)} = \tilde{c}|_G$ is the transgression of the cocycle $c_{(\bullet)}$.

Applying this to the constructed above cocycle $c_3^{(v)}$ we get a current $w_v^{3,2} \in \mathcal{A}^5_{GL_3(\mathbb{C})}$. It is easy to check that it defines a nontrivial class in $H^5_{\text{top}}(GL_3(\mathbb{C}))$. So the cocycle $c_3^{(v)}$ represents a nontrivial nondecomposable class in $H^6_{\mathcal{D}}(BGL_3(\mathbb{C})_\bullet, R(3))$ □

Theorem 5.11. *The 5-cocycle* $\mathcal{L}_3(r_3(v, g_1v, \ldots, g_5v))$ *defines a nontrivial class in* $H^5_{\text{cts}}(GL_3(\mathbb{C}), R)$.

Proof. Let G^δ be the Lie group made discrete. The morphism of groups $GL_3(\mathbb{C})^\delta \longrightarrow GL_3(\mathbb{C})$ provides a morphism

$$e : BGL_3(\mathbb{C})^\delta_\bullet \longrightarrow BGL_3(\mathbb{C})_\bullet\ .$$

Therefore

$$\begin{array}{rcl} e^* & : & H^6_{\mathcal{D}}(BGL_3(\mathbb{C})_\bullet, R(3)) \longrightarrow H^6_{\mathcal{D}}(BGL_3(\mathbb{C})^\delta, R(3)) = \\ & & = H^5(BGL_3(\mathbb{C})_\bullet, S^0) \equiv H^5_{\text{cts}}(GL_3(\mathbb{C}), R) \end{array}$$

(S^0 is the sheaf of C^∞-functions). It is known that e^* maps the indecomposable class in $H^6_{\mathcal{D}}(BGL_3(\mathbb{C})_\bullet, R(3))$ just to non zero multiple of the Borel class in $H^5_{\text{cts}}(GL_3(\mathbb{C}), R)$. (This is a particular case of the Beilinson's theorem comparing his regulator with the Borel one). In our case $e^*(c_3^{(v)}) = \mathcal{L}_3(r_3(v, g_1v, \ldots, g_5v))$ by construction. □

5. Possible generalizations. Recall that $(T_*(n), \partial)$ is the total complex associated with the Grassmanian bicomplex (3.18) and $T_{n+1}(n) = C_{n+1}(n)$.

Optimistic Conjecture 5.11. *There exists a homomorphism of complexes* $\psi_*(n)$:

$$\begin{array}{ccccccc} \xrightarrow{\partial} & T_{2n}(n) & \xrightarrow{\partial} & \dots \longrightarrow & T_{n+2}(n) & \xrightarrow{\partial} & T_{n+1}(n) \\ & \downarrow {\scriptstyle \psi_{2n}(n)} & & & \downarrow {\scriptstyle \psi_{n+2}(n)} & & \downarrow {\scriptstyle \psi_{n+1}(n)} \\ 0 \longrightarrow & \mathcal{B}_n(F) & \xrightarrow{\delta} & \dots \xrightarrow{\delta} & \mathcal{B}_2(F)\otimes\wedge^{n-2}F^* & \xrightarrow{\delta} & \wedge^n F^* \end{array}$$

such that

$$\psi_{n+1}(n) : (l_0, \dots, l_n) \in C_{n+1}(n) \mapsto \text{Alt} \wedge_{i=1}^{n} \Delta(l_0, \dots, \hat{l}_i, \dots, l_n) \in \wedge^n F^* \ .$$

This conjecture together with formulas for $\psi_*(n)$ imply all explicit formulas for characteristic classes that I can imagine. Let me illustrate this by the following examples.

Corollary 5.12. *Conjecture 5.11 imply a construction of the Chern classes*

$$C_{i,n} : K_{2n-i}^{[n-i]}(F)_{\mathbb{Q}} \longrightarrow H^i(\Gamma_F(n)_{\mathbb{Q}})$$

(I use the rank filtration instread of the Adams one).

Proof. See s. 7,10 in §3. □

Corollary 5.13. *Zagier's conjecture about* $\zeta_F(n)$ *follows from conjecture 5.11.*

Proof. For $n = 3$ this was explained in s. 7,10 in §3 and §5. See [G4] for general case □

The function $P_n := \tilde{\mathcal{L}}_n \circ \psi_{2n}(n)$ on $C_{2n}(n)$

$$P_n : (l_0, \dots, l_{2n-1}) \overset{\psi_{2n}(n)}{\longrightarrow} \mathcal{B}_n(\mathbb{C}) \overset{\tilde{\mathcal{L}}_n}{\longrightarrow} R$$

satisfies the functional equations

$$\sum_{i=0}^{2n} (-1)^i P_n(l_0, \dots, \hat{l}_i, \dots, l_{2n}) = 0 \quad \forall (l_0, \dots, l_{2n}) \in C_{2n+1}(n)$$

$$\sum_{i=}^{2n} (-1)^i P_n(l_i | l_0, \dots, \hat{l}_i, \dots, l_{2n}) = 0 \quad \forall (l_0, \dots, l_{2n}) \in C_{2n+1}(n+1) \ .$$

Therefore for a nonzero vector $v \in \mathbb{C}^n$ the function $P_n(v, g_1 v, \quad, g_{2n} v)$ is a measurable $(2n-1)$-cocycle of $GL_n(\mathbb{C})$ representing the Borel class in $H^{2n-1}_{cts}(GL_n(\mathbb{C}), R)$. (For a genearlization of this construction to $N > n$ see [G4]).

Formulas for $\psi_*(n)$ provide an explicit construction of the universal Chern class $c_n \in H^{2n}_{\mathcal{M}}(BGL_N(F)_\bullet, \mathbb{Q}(n))$, $(N \geq n)$, together with their realization in Deligne cohomology. In particular we will get an explicit construction of the Chern classes of vector bundles with values in motivic cohomology (see [G4]). I would like to emphasize that all this is closely related to the work of Gabrielov, Gelfand and Losik about combinatorial formula for the first Pontryagin class ([GGL], [You]).

The Grassmanian complex $(C_*(n), d)$ is a subcomplex in $(T_*(n), \partial)$. Therefore homomorphism $\psi_*(n)$ provides a formula for the Grassmanian n-cocycle in Deligne cohomology conjectured in [BMS], [HM].

It is interesting that for applications (to characteristic classes for instance) it is *not* sufficient to have such formulas for the Grassmanian complex only: we have to extend them to the whole Grassmanian *bicomplex*. This problem becomes nontrivial already for $n = 4$.

Another important application of formulas for $\psi_*(n)$ is a very explicit construction using the classical polylogarithms for Beilinson's regulator for curves and, moreover, arbitrary regular schemes X. Together with Beilinson's conjecture about regulators this will give us an (hypothetical) explicit formula for $\zeta_X(n)$. Note that such formulas can be written without mentioning conjecture 5.11, see [G3].

Today I know an explicit formula (for arbitrary n) for $\psi_{n+2}(n)$ and $\psi_{n+1}(n)$ only. I think that formulas for $\psi_*(n)$ are the priority problem. For $n = 2, 3$ this was done in [G2], but theorem 4.7 indicates that unexpected phenomenas can appear for $n \geq 4$. The case $n = 4$ is crucial for understanding whether conjecture 5.11 is true or not, and it will be certainly quite different from $n = 2, 3$.

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